LOCALIZATION OF MODULES OVER SMALL QUANTUM GROUPS

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1. Introduction

This article is a report on [FS], [BFS].

1.1. Let $(I, \cdot)$ be a Cartan datum of finite type and $(Y, X, \dots)$ the simply connected root datum of type $(I, \cdot)$, cf. [L1].

Let $l > 1$ be an integer. Set $\ell = l/(l, 2)$; to simplify the exposition, we will suppose in this Introduction that $d_i := i \cdot i/2$ divides ℓ for all $i \in I$; we set $\ell_i := \ell/d_i$. We suppose that all $\ell_i > 3$. Let $\rho \in X$ be the half-sum of positive roots; let $\rho_\ell \in X$ be defined by $\langle i, \rho_\ell\rangle = \ell_i - 1$ $(i \in I)$. We define a lattice $Y_\ell \subset Y$ by $Y_\ell = \{\mu \in X|$ for all $\mu' \in X, \mu \cdot \mu' \in \ell\mathbb{Z}\}$, and set $\mathrm{d}_\ell = \mathrm{card}(X/Y_\ell)$.

We fix a base field k of characteristic not dividing l. We suppose that k contains a primitive l-th root of unity ζ, and fix it. Starting from these data, one defines certain category $\mathcal{C}$. Its objects are finite dimensional X-graded k-vector spaces equipped with an action of Lusztig's "small" quantum group $\mathfrak{u}$ (cf. [L2]) such that the action of its Cartan subalgebra is compatible with the X-grading. Variant: one defines certain algebra $\dot{\mathfrak{u}}$ which is an "X-graded" version of $\mathfrak{u}$ (see 12.2), and an object of $\mathcal{C}$ is a finite dimensional $\dot{\mathfrak{u}}$-module. For the precise definition of $\mathcal{C}$, see 2.11, 2.13. For l prime and $\mathrm{char} k = 0$, the category $\mathcal{C}$ was studied in [AJS], for $\mathrm{char} k > 0$ and arbitrary l, $\mathcal{C}$ was studied in [AW]. The category $\mathcal{C}$ admits a remarkable structure of a *ribbon*[1] category (Lusztig).

1.2. The main aim of this work is to introduce certain tensor category $\mathcal{FS}$ of geometric origin, which is equivalent to $\mathcal{C}$. Objects of $\mathcal{FS}$ are called **(finite) factorizable sheaves**.

A notion of a factorizable sheaf is the main new concept of this paper. Let us give an informal idea, what kind of an object is it. Let D be the unit disk in the complex plane[2].

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[1] in other terminology, braided balanced rigid tensor

[2] One could also take a complex affine line or a formal disk; after a suitable modification of the definitions, the resulting categories of factorizable sheaves are canonically equivalent.

Let $\mathcal{D}$ denote the space of positive Y-valued divisors on D. Its points are formal linear combinations $\sum \nu \cdot x$ ($\nu \in Y^+ := \mathbb{N}[I], x \in D$). This space is a disjoint union

$$\mathcal{D} = \coprod_{\nu \in Y^+} D^\nu$$

where D^ν is the subspace of divisors of degree ν. Variant: D^ν is the configuration space of ν points running on D; its points are (isomorphism clases of) pairs of maps

$(J \longrightarrow D,\, j \mapsto x_j;\, \pi : J \longrightarrow I,\, j \mapsto i_j)$,

J being a finite set. We say that we have a finite set $\{x_j\}$ of (possibly equal) points of D, a point x_j being "coloured" by the colour i_j. The sum (in Y) of colours of all points should be equal to ν. The space D^ν is a smooth affine analytic variety; it carries a canonical stratification defined by various intersections of hypersurfaces $\{x_j = 0\}$ and $\{x_{j'} = x_{j''}\}$. The open stratum (the complement of all the hypersurfaces above) is denoted by $A^{\nu\circ}$.

One can imagine $\mathcal{D}$ as a Y^+-graded smooth stratified variety. Let $\mathcal{A} = \coprod A^\nu \subset \mathcal{D}$ be the open Y^+-graded subvariety of positive Y-valued divisors on $D - \{0\}$. We have an open Y^+-graded subvariety $\mathcal{A}^\circ = \coprod A^{\nu\circ} \subset \mathcal{A}$.

Let us consider the $(Y^+)^2$-graded variety

$$\mathcal{A} \times \mathcal{A} = \coprod A^{\nu_1} \times A^{\nu_2};$$

we define another $(Y^+)^2$-graded variety $\widetilde{\mathcal{A} \times \mathcal{A}}$, together with two maps

(a) $\mathcal{A} \times \mathcal{A} \xleftarrow{p} \widetilde{\mathcal{A} \times \mathcal{A}} \xrightarrow{m} \mathcal{A}$

respecting the Y^+-gradings[3]; the map p is a homotopy equivalence. One can imagine the diagram (a) above as a "homotopy multiplication"

$$m_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A};$$

this "homotopy map" is "homotopy associative"; the meaning of this is explained in the main text. We say that $\mathcal{A}$ is a (Y^+-graded) "homotopy monoid"; $\mathcal{A}^\circ \subset \mathcal{A}$ is a "homotopy submonoid".

The space $\mathcal{D}$ is a "homotopy $\mathcal{A}$-space": there is a "homotopy map"

$$m_{\mathcal{D}} : \mathcal{D} \times \mathcal{A} \longrightarrow \mathcal{D}$$

which is, as above, a diagram of usual maps between Y^+-graded varieties

(b) $\mathcal{D} \times \mathcal{A} \xleftarrow{p} \widetilde{\mathcal{D} \times \mathcal{A}} \xrightarrow{m} \mathcal{D}$,

p being a homotopy equivalence.

For each $\mu \in X,\ \nu \in Y^+$, one defines a one-dimensional k-local system $\mathcal{I}^\nu_\mu$ over $A^{\nu\circ}$. Its monodromies are defined by variuos scalar products of "colours" of running points and

[3] a $(Y^+)^2$-graded space is considered as a Y^+-graded by means of the addition $(Y^+)^2 \longrightarrow Y^+$.

the colour μ of the origin $0 \in D$. These local systems have the following compatibility. For each $\mu \in X, \nu_1, \nu_2 \in Y^+$, a *factorization isomorphism*

$$\phi_\mu(\nu_1,\nu_2) : m^*\mathcal{I}_\mu^{\nu_1+\nu_2} \xrightarrow{\sim} p^*(\mathcal{I}_\mu^{\nu_1} \boxtimes \mathcal{I}_{\mu-\nu_1}^{\nu_2})$$

is given (where m, p are the maps in the diagram (a) above), these isomorphisms satisfying certain *(co)associativity property*. The collection of local systems $\{\mathcal{I}_\mu^\nu\}$ is an X-graded local system $\mathcal{I} = \oplus\, \mathcal{I}_\mu$ over $\mathcal{A}^\circ$. One could imagine the collection of factorization isomorphisms $\{\phi_\mu(\nu_1,\nu_2)\}$ as an isomorphism

$$\phi : \ m^*_{\mathcal{A}^\circ}\mathcal{I} \xrightarrow{\sim} \mathcal{I} \boxtimes \mathcal{I}$$

We call $\mathcal{I}$ the *braiding local system.*

Let $\mathcal{I}^\bullet$ be the perverse sheaf on $\mathcal{A}$ which is the Goresky-MacPherson extension of the local system $\mathcal{I}$. The isomorphism ϕ above induces a similar isomorphism

$$\phi^\bullet : \ m^*_{\mathcal{A}}\mathcal{I}^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet \boxtimes \mathcal{I}^\bullet$$

This sheaf, together with $\phi^\bullet$, looks like a "coalgebra"; it is an incarnation of the quantum group $\overset{\bullet}{\mathfrak{u}}$.

A factorizable sheaf is a couple

($\mu \in X$ ("the highest weight"); a perverse sheaf $\mathcal{X}$ over $\mathcal{D}$, smooth along the canonical stratification).

Thus, $\mathcal{X}$ is a collection of sheaves $\mathcal{X}^\nu$ over $\mathcal{D}^\nu$. These sheaves should be connected by *factorization isomorphisms*

$$\psi(\nu_1,\nu_2) : \ m^*\mathcal{X}^{\nu_1+\nu_2} \xrightarrow{\sim} p^*(\mathcal{X}^{\nu_1} \boxtimes \mathcal{I}_{\mu-\nu_1}^{\nu_2\bullet})$$

satisfying an associativity property. Here m, p are as in the diagram (b) above. One could imagine the whole collection $\{\psi(\nu_1,\nu_2)\}$ as an isomorphism

$$\psi : m^*_{\mathcal{D}}\mathcal{X} \xrightarrow{\sim} \mathcal{I}^\bullet \boxtimes \mathcal{X}$$

satisfying a (co)associativity property. We impose also certain finiteness (of singularities) condition on $\mathcal{X}$. So, this object looks like a "comodule" over $\mathcal{I}^\bullet$. It is an incarnation of an $\overset{\bullet}{\mathfrak{u}}$-module.

We should mention one more important part of the structure of the space $\mathcal{D}$. It comes from natural closed embeddings $\iota_\nu : \ \mathcal{D} \hookrightarrow \mathcal{D}[\nu]$ (were $[\nu]$ denotes the shift of the grading); these mappings define certain inductive system, and a factorizable sheaf is a sheaf on its inductive limit.

The latter inductive limit is an example of a **"semiinfinite space"**.

For the precise definitions, see Sections 3, 4 below.

The category $\mathcal{FS}$ has a structure of a braided balanced tensor category coming from geometry. The tensor structure on $\mathcal{FS}$ is defined using the functors of nearby cycles. The

tensor equivalence

$$\Phi : \mathcal{FS} \xrightarrow{\sim} \mathcal{C}$$

is defined using vanishing cycles functors. It respects the braidings and balances. This is the contents of Part I.

1.3. Factorizable sheaves are local objects. It turns out that one can "glue" them along complex curves. More precisely, given a finite family $\{\mathcal{X}_a\}_{a\in A}$ of factorizable sheaves and a smooth proper curve C together with a family $\{x_a, \tau_a\}_A$ of distinct points $x_a \in C$ with non-zero tangent vectors τ_a at them (or, more generally, a family of such objects over a smooth base S), one can define a perverse sheaf, denoted by

$$\boxed{\times}_A^{(C)} \mathcal{X}_a$$

on the (relative) configuration space C^ν. Here $\nu \in Y^+$ is defined by

(a) $\nu = \sum_{a\in A} \mu_a + (2g-2)\rho_\ell$

where g is the genus of C, μ_a is the highest weight of $\mathcal{X}_a$. We *assume* that the right hand side of the equality belongs to Y^+.

One can imagine this sheaf as an "exterior tensor product" of the family $\{\mathcal{X}_a\}$ along C. It is obtained by "planting" the sheaves $\mathcal{X}_a$ into the points x_a. To glue them together, one needs a "glue". This glue is called the **Heisenberg local system** $\mathcal{H}$; it was constructed by R.Bezrukavnikov. $\mathcal{H}$ is a sister of the braiding local system $\mathcal{I}$. Let us describe what it is.

For a finite set A, let $\mathcal{M}_A$ denote the moduli stack of punctured curves $(C, \{x_a, \tau_a\}_A)$ as above; let $\eta : C_A \longrightarrow \mathcal{M}_A$ be the universal curve. Let $\lambda_A = \det(R\eta_*\mathcal{O}_{C_A})$ be the determinant line bundle on $\mathcal{M}_A$, and $\mathcal{M}_{A;\lambda} \longrightarrow \mathcal{M}_A$ be the total space of λ_A with the zero section removed. For $\nu \in Y^+$, let $\eta^\nu : C_A^\nu \longrightarrow \mathcal{M}_A$ be the relative configuration space of ν points running on C_A; let $C_A^{\nu\circ} \subset C_A^\nu$ be the open stratum (where the running points are distinct and distinct from the punctures x_a). The complementary subscript $()_\lambda$ will denote the base change under $\mathcal{M}_{A;\lambda} \longrightarrow \mathcal{M}_A$. The complementary subscript $()_g$ will denote the base change under $\mathcal{M}_{A,g} \hookrightarrow \mathcal{M}_A$, $\mathcal{M}_{A,g}$ being the substack of curves of genus g.

The Heisenberg local system is a collection of local systems $\mathcal{H}_{\vec{\mu};A,g}$ over the stacks $C^{\nu\circ}_{A,g;\lambda}$. Here $\vec{\mu}$ is an A-tuple $\{\mu_a\} \in X^a$; $\nu = \nu(\vec{\mu}; g)$ is defined by the equality (a) above. We assume that $\vec{\mu}$ is such that the right hand side of this equality really belongs to Y^+. The dimension of $\mathcal{H}_{\vec{\mu};A,g}$ is equal to d_ℓ^g; the monodromy around the zero section of the determinant line bundle is equal to

(b) $c = (-1)^{\mathrm{card}(I)}\zeta^{-12\rho\cdot\rho}$.

This formula is due to R.Bezrukavnikov. These local systems have a remarkable compatibility ("fusion") property which we do not specify here, see Section 15.

1.4. In Part II we study the sheaves $\boxed{\times}_A^{(C)}\ \mathcal{X}_a$ for $C = \mathbb{P}^1$. Their cohomology, when expressed algebraically, turn out to coincide with certain "semiinfinite" Tor spaces in the category $\mathcal{C}$ introduced by S.M.Arkhipov. Due to the results of Arkhipov, this enables one to prove that "spaces of conformal blocks" in WZW models are the natural subquotients of such cohomology spaces.

1.5. In Part III we study the sheaves $\boxed{\times}_A^{(C)}\ \mathcal{X}_a$ for arbitrary smooth families of punctured curves. Let $\boxed{\times}_{A,g}\ \mathcal{X}_a$ denotes the "universal exterior product" living on $C^{\nu}_{A,g;\lambda}$ where $\nu = \nu(\vec{\mu}; g)$ is as in 1.3 (a) above, $\vec{\mu} = \{\mu_a\}$, μ_a being the highest weight of $\mathcal{X}_a$. Let us integrate it: consider

$$\int_{C^\nu} \boxed{\times}_{A,g}\ \mathcal{X}_a := R\eta^{\nu}_{*}(\boxed{\times}_{A,g}\ \mathcal{X}_a);$$

it is a complex of sheaves over $\mathcal{M}_{A,g;\lambda}$ with smooth cohomology; let us denote it $\langle \otimes_A\ \mathcal{X}_a\rangle_g$. The cohomology sheaves of such complexes define a *fusion structure*[4] on $\mathcal{FS}$ (and hence on $\mathcal{C}$).

The classical WZW model fusion category is, in a certain sense, a subquotient of one of them. The number c, 1.3 (b), coincides with the "multiplicative central charge" of the model (in the sense of [BFM]).

As a consequence of this geometric description and of the Purity Theorem, [BBD], the local systems of conformal blocks (in arbitrary genus) are semisimple. The Verdier duality induces a canonical non-degenerate Hermitian form on them (if $k = \mathbb{C}$).

Almost all the results of Part III are due to R.Bezrukavnikov, cf. [BFS].

1.6. The works [FS] and [BFS] in a sense complete the program outlined in [S] (no fusion structure in higher genus was mentioned there). We should mention a very interesting related work of G.Felder and collaborators. The idea of realizing quantum groups' modules in the cohomology of configuration spaces appeared independently in [FW]. The Part III of the present work arose from our attempts to understand [CFW].

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1.7. This article contains almost all main definitions and theorems from [FS], [BFS], but no proofs; we refer the reader to *op. cit.* for them. However, our exposition here differs from *op. cit.*; we hope it clarifies the subject to some extent.

[4] actually a family of such structures (depending on the degree of cohomology)

Part I. Local.

2. The category $\mathcal{C}$

2.1. We will follow Lusztig's terminology and notations concerning root systems, cf. [L1].

We fix an irreducible Cartan datum $(I, \cdot)$ of finite type. Thus, I is a finite set together with a symmetric $\mathbb{Z}$-valued bilinear form $\nu_1, \nu_2 \mapsto \nu_1 \cdot \nu_2$ on the free abelian group $\mathbb{Z}[I]$, such that for any $i \in I$, $i \cdot i$ is even and positive, for any $i \neq j$ in I, $2i \cdot j/i \cdot i \leq 0$ and the $I \times I$-matrix $(i \cdot j)$ is positive definite. We set $d_i = i \cdot i/2$.

Let $d = \max_{i \in I}\ d_i$. This number is equal to the least common multiple of the numbers d_i, and belongs to the set $\{1, 2, 3\}$. For a simply laced root datum $d = 1$.

We set $Y = \mathbb{Z}[I], X = \mathrm{Hom}(Y, \mathbb{Z})$; $\langle , \rangle :\ Y \times X \longrightarrow \mathbb{Z}$ will denote the obvious pairing. The obvious embedding $I \hookrightarrow Y$ will be denoted by $i \mapsto i$. We will denote by $i \mapsto i'$ the embedding $I \hookrightarrow X$ given by $\langle i, j' \rangle = 2i \cdot j/i \cdot i$. (Thus, $(Y, X, \ldots)$ is the simply connected root datum of type $(I, \cdot)$, in the terminology of Lusztig.)

The above embedding $I \subset X$ extends by additivity to the embedding $Y \subset X$. We will regard Y as the sublattice of X by means of this embedding. For $\nu \in Y$, we will denote by the same letter ν its image in X. We set $Y^+ = \mathbb{N}[I] \subset Y$.

We will use the following partial order on X. For $\mu_1, \mu_2 \in X$, we write $\mu_1 \leq \mu_2$ if $\mu_2 - \mu_1$ belongs to Y^+.

2.2. We fix a base field k, an integer $l > 1$ and a primitive root of unity $\zeta \in k$ as in the Introduction.

We set $\ell = l$ if l is odd and $\ell = l/2$ if l is even. For $i \in I$, we set $\ell_i = \ell/(\ell, d_i)$. Here (a, b) stands for the greatest common divisor of a, b. We set $\zeta_i = \zeta^{d_i}$. We will assume that $\ell_i > 1$ for any $i \in I$ and $\ell_i > -\langle i, j' \rangle + 1$ for any $i \neq j$ in I.

We denote by ρ (resp. ρ_ℓ) the element of X such that $\langle i, \rho \rangle = 1$ (resp. $\langle i, \rho_\ell \rangle = \ell_i - 1$) for all $i \in I$.

For a coroot $\beta \in Y$, there exists an element w of the Weyl group W of our Cartan datum and $i \in I$ such that $w(i) = \beta$. We set $\ell_\beta := \frac{\ell}{(\ell, d_i)}$; this number does not depend on the choice of w and i.

We have $\rho = \frac{1}{2} \sum\ \alpha$; $\rho_\ell = \frac{1}{2} \sum\ (\ell_\alpha - 1)\alpha$, the sums over all positive roots $\alpha \in X$.

For $a \in \mathbb{Z}, i \in I$, we set $[a]_i = 1 - \zeta_i^{-2a}$.

2.3. We use the same notation $\mu_1, \mu_2 \mapsto \mu_1 \cdot \mu_2$ for the unique extension of the bilinear form on Y to a $\mathbb{Q}$-valued bilinear form on $Y \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{Q}$.

We define a lattice $Y_\ell = \{\lambda \in X |$ for all $\mu \in X, \lambda \cdot \mu \in \ell\mathbb{Z}\}$.

2.4. Unless specified otherwise, a "vector space" will mean a vector space over k; $\otimes$ will denote the tensor product over k. A "sheaf" (or a "local system") will mean a sheaf (resp. local system) of k-vector spaces.

If $(T, \mathcal{S})$ is an open subspace of the space of complex points of a separate scheme of finite type over $\mathbb{C}$, with the usual topology, together with an algebraic stratification $\mathcal{S}$ satisfying the properties [BBD] 2.1.13 b), c), we will denote by $\mathcal{M}(T; \mathcal{S})$ the category of perverse sheaves over T lisse along $\mathcal{S}$, with respect to the middle perversity, cf. [BBD] 2.1.13, 2.1.16.

2.5. Let $'\mathfrak{f}$ be the free associative k-algebra with 1 with generators θ_i $(i \in I)$. For $\nu = \sum \nu_i i \in \mathbb{N}[I]$, let $'\mathfrak{f}_\nu$ be the subspace of $'\mathfrak{f}$ spanned by the monomials $\theta_{i_1} \cdot \ldots \cdot \theta_{i_a}$ such that $\sum_j i_j = \nu$ in $\mathbb{N}[I]$.

Let us regard $'\mathfrak{f} \otimes {}'\mathfrak{f}$ as a k-algebra with the product $(x_1 \otimes x_2)(y_1 \otimes y_2) = \zeta^{\nu \cdot \mu} x_1 y_1 \otimes x_2 y_2$ $(x_2 \in {}'\mathfrak{f}_\nu, y_1 \in {}'\mathfrak{f}_\mu)$. Let r denote a unique homomorphism of k-algebras $'\mathfrak{f} \longrightarrow {}'\mathfrak{f} \otimes {}'\mathfrak{f}$ carrying θ_i to $1 \otimes \theta_i + \theta_i \otimes 1$ $(i \in I)$.

2.6. **Lemma-definition.** *There exists a unique k-valued bilinear form $(\cdot, \cdot)$ on $'\mathfrak{f}$ such that*

(i) $(1,1) = 1;$ $(\theta_i, \theta_j) = \delta_{ij}$ $(i, j \in I)$*;* (ii) $(x, yy') = (r(x), y \otimes y')$ *for all* $x, y, y' \in {}'\mathfrak{f}$.

This bilinear form is symmetric. $\square$

In the right hand side of the equality (ii) we use the same notation $(\cdot, \cdot)$ for the bilinear form on $'\mathfrak{f} \otimes {}'\mathfrak{f}$ defined by $(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2, y_2)$.

The radical of the form $'\mathfrak{f}$ is a two-sided ideal of $'\mathfrak{f}$.

2.7. Let us consider the associative k-algebra $\mathfrak{u}$ (with 1) defined by the generators ϵ_i, θ_i $(i \in I), K_\nu$ $(\nu \in Y)$ and the relations (a) — (e) below.

(a) $K_0 = 1,$ $K_\nu \cdot K_\mu = K_{\nu+\mu}$ $(\nu, \mu \in Y)$;

(b) $K_\nu \epsilon_i = \zeta^{\langle \nu, i' \rangle} \epsilon_i K_\nu$ $(i \in I, \nu \in Y)$;

(c) $K_\nu \theta_i = \zeta^{-\langle \nu, i' \rangle} \theta_i K_\nu$ $(i \in I, \nu \in Y)$;

(d) $\epsilon_i \theta_j - \zeta^{i \cdot j} \theta_j \epsilon_i = \delta_{ij}(1 - \tilde{K}_i^{-2})$ $(i, j \in I)$.

Here we use the notation $\tilde{K}_\nu = \prod_i K_{d_i \nu_i i}$ $(\nu = \sum \nu_i i)$.

(e) If $f(\theta_i) \in {}'\mathfrak{f}$ belongs to the radical of the form $(\cdot, \cdot)$ then $f(\theta_i) = f(\epsilon_i) = 0$ in $\mathfrak{u}$.

2.8. There is a unique k-algebra homomorphism $\Delta : \mathfrak{u} \longrightarrow \mathfrak{u} \otimes \mathfrak{u}$ such that

$$\Delta(\epsilon_i) = \epsilon_i \otimes 1 + \tilde{K}_i^{-1} \otimes \epsilon_i;\ \Delta(\theta_i) = \theta_i \otimes 1 + \tilde{K}_i^{-1} \otimes \theta_i;\ \Delta(K_\nu) = K_\nu \otimes K_\nu$$

for any $i \in I, \nu \in Y$. Here $\mathfrak{u} \otimes \mathfrak{u}$ is regarded as an algebra in the standard way.

There is a unique k-algebra homomorphism $e : \mathfrak{u} \longrightarrow k$ such that $e(\epsilon_i) = e(\theta_i) = 0, e(K_\nu) = 1$ $(i \in I, \nu \in Y)$.

2.9. There is a unique k-algebra homomorphism $A : \mathfrak{u} \longrightarrow \mathfrak{u}^{\mathrm{opp}}$ such that

$$A(\epsilon_i) = -\epsilon_i \tilde{K}_i;\ A(\theta_i) = -\theta_i \tilde{K}_i;\ A(K_\nu) = K_{-\nu}\ (i \in I, \nu \in Y).$$

There is a unique k-algebra homomorphism $A' : \mathfrak{u} \longrightarrow \mathfrak{u}^{\mathrm{opp}}$ such that

$$A'(\epsilon_i) = -\tilde{K}_i \epsilon_i;\ A'(\theta_i) = -\tilde{K}_i \theta_i;\ A'(K_\nu) = K_{-\nu}\ (i \in I, \nu \in Y).$$

2.10. The algebra $\mathfrak{u}$ together with the additional structure given by the comultiplication Δ, the counit e, the antipode A and the skew-antipode A', is a Hopf algebra.

2.11. Let us define a category $\mathcal{C}$ as follows. An object of $\mathcal{C}$ is a $\mathfrak{u}$-module M which is finite dimensional over k, with a given direct sum decomposition $M = \oplus_{\lambda \in X} M_\lambda$ (as a vector space) such that $K_\nu x = \zeta^{\langle \nu, \lambda \rangle} x$ for any $\nu \in Y, \lambda \in X, x \in M_\lambda$. A morphism in $\mathcal{C}$ is a $\mathfrak{u}$-linear map respecting the X-gradings.

Alternatively, an object of $\mathcal{C}$ may be defined as an X-graded finite dimensional vector space $M = \oplus M_\lambda$ equipped with linear operators

$$\theta_i : M_\lambda \longrightarrow M_{\lambda - i'},\ \epsilon_i : M_\lambda \longrightarrow M_{\lambda + i'}\ (i \in I, \lambda \in X)$$

such that

(a) for any $i, j \in I, \lambda \in X$, the operator $\epsilon_i \theta_j - \zeta^{i \cdot j} \theta_j \epsilon_i$ acts as the multiplication by $\delta_{ij} [\langle i, \lambda \rangle]_i$ on M_λ.

Note that $[\langle i, \lambda \rangle]_i = [\langle d_i i, \lambda \rangle] = [i' \cdot \lambda]$.

(b) If $f(\theta_i) \in {}'\mathfrak{f}$ belongs to the radical of the form $(\cdot, \cdot)$ then the operators $f(\theta_i)$ and $f(\epsilon_i)$ act as zero on M.

2.12. In [L2] Lusztig defines an algebra $\mathbf{u}_\mathcal{B}$ over the ring $\mathcal{B}$ which is a quotient of $\mathbb{Z}[v, v^{-1}]$ (v being an indeterminate) by the l-th cyclotomic polynomial. Let us consider the k-algebra $\mathbf{u}_k$ obtained from $\mathbf{u}_\mathcal{B}$ by the base change $\mathcal{B} \longrightarrow k$ sending v to ζ. The algebra $\mathbf{u}_k$ is generated by certain elements E_i, F_i, K_i $(i \in I)$. Here $E_i = E_{\alpha_i}^{(1)}, F_i = F_{\alpha_i}^{(1)}$ in the notations of *loc.cit.*

Given an object $M \in \mathcal{C}$, let us introduce the operators E_i, F_i, K_i on it by

$$E_i = \frac{\zeta_i}{1 - \zeta_i^{-2}} \epsilon_i \tilde{K}_i,\ F_i = \theta_i, K_i = K_i.$$

2.13. **Theorem.** *The above formulas define the action of the Lusztig's algebra* $\mathbf{u}_k$ *on an object* M.

This rule defines an equivalence of $\mathcal{C}$ *with the category whose objects are* X*-graded finite dimensional* $\mathbf{u}_k$*-modules* $M = \oplus M_\lambda$ *such that* $K_i x = \zeta^{\langle i,\lambda\rangle} x$ *for any* $i \in I, \lambda \in X, x \in M_\lambda$. $\square$

2.14. The structure of a Hopf algebra on $\mathfrak{u}$ defines canonically a *rigid tensor* structure on $\mathcal{C}$ (cf. [KL]IV, Appendix).

The Lusztig's algebra $\mathbf{u}_k$ also has an additional structure of a Hopf algebra. It induces the same rigid tensor structure on $\mathcal{C}$.

We will denote the duality in $\mathcal{C}$ by $M \mapsto M^*$. The unit object will be denoted by $\mathbf{1}$.

2.15. Let $\mathfrak{u}^-$ (resp. $\mathfrak{u}^+$, $\mathfrak{u}^0$) denote the k-subalgebra generated by the elements θ_i $(i \in I)$ (resp. ϵ_i $(i \in I)$, K_ν $(\nu \in Y)$). We have the triangular decomposition $\mathfrak{u} = \mathfrak{u}^-\mathfrak{u}^0\mathfrak{u}^+ = \mathfrak{u}^+\mathfrak{u}^0\mathfrak{u}^-$.

We define the "Borel" subalgebras $\mathfrak{u}^{\leq 0} = \mathfrak{u}^-\mathfrak{u}^0$, $\mathfrak{u}^{\geq 0} = \mathfrak{u}^+\mathfrak{u}^0$; they are the Hopf subalgebras of $\mathfrak{u}$.

Let us introduce the X-grading $\mathfrak{u} = \oplus \mathfrak{u}_\lambda$ as a unique grading compatible with the structure of an algebra such that $\theta_i \in \mathfrak{u}_{-i'}$, $\epsilon_i \in \mathfrak{u}_{i'}$, $K_\nu \in \mathfrak{u}_0$. We will use the induced gradings on the subalgebras of $\mathfrak{u}$.

2.16. Let $\mathcal{C}^{\leq 0}$ (resp. $\mathcal{C}^{\geq 0}$) be the category whose objects are X-graded finite dimensional $\mathfrak{u}^{\leq 0}$- (resp. $\mathfrak{u}^{\geq 0}$-) modules $M = \oplus M_\lambda$ such that $K_\nu x = \zeta^{\langle \nu,\lambda\rangle} x$ for any $\nu \in Y, \lambda \in X, x \in M_\lambda$. Morphisms are $\mathfrak{u}^{\leq 0}$- (resp. $\mathfrak{u}^{\geq 0}$-) linear maps compatible with the X-gradings.

We have the obvious functors $\mathcal{C} \longrightarrow \mathcal{C}^{\leq 0}$ (resp. $\mathcal{C} \longrightarrow \mathcal{C}^{\geq 0}$). These functors admit the exact left adjoints $\mathrm{ind}_{\mathfrak{u}^{\leq 0}}^{\mathfrak{u}} : \mathcal{C}^{\leq 0} \longrightarrow \mathcal{C}$ (resp. $\mathrm{ind}_{\mathfrak{u}^{\geq 0}}^{\mathfrak{u}} : \mathcal{C}^{\geq 0} \longrightarrow \mathcal{C}$).

For example, $\mathrm{ind}_{\mathfrak{u}^{\geq 0}}^{\mathfrak{u}}(M) = \mathfrak{u} \otimes_{\mathfrak{u}^{\geq 0}} M$. The triangular decomposition induces an isomorphism of graded vector spaces $\mathrm{ind}_{\mathfrak{u}^{\geq 0}}^{\mathfrak{u}}(M) \cong \mathfrak{u}^- \otimes M$.

2.17. For $\lambda \in X$, let us consider on object $k^\lambda \in \mathcal{C}^{\geq 0}$ defined as follows. As a graded vector space, $k^\lambda = k^\lambda_\lambda = k$. The algebra $\mathfrak{u}^{\geq 0}$ acts on k^λ as follows: $\epsilon_i x = 0, K_\nu x = \zeta^{\langle \nu,\lambda\rangle} x$ $(i \in I, \nu \in Y, x \in k^\lambda)$.

The object $M(\lambda) = \mathrm{ind}_{\mathfrak{u}^{\geq 0}}^{\mathfrak{u}}(k^\lambda)$ is called a *(baby) Verma module*. Each $M(\lambda)$ has a unique irreducible quotient object, to be denoted by $L(\lambda)$. The objects $L(\lambda)$ $(\lambda \in X)$ are mutually non-isomorphic and every irreducible object in $\mathcal{C}$ is isomorphic to one of them. Note that the category $\mathcal{C}$ is *artinian*, i.e. each object of $\mathcal{C}$ has a finite filtration with irreducible quotients.

For example, $L(0) = \mathbf{1}$.

2.18. Recall that a *braiding* on $\mathcal{C}$ is a collection of isomorphisms

$$R_{M,M'} : M \otimes M' \xrightarrow{\sim} M' \otimes M \ (M, M' \in \mathcal{C})$$

satisfying certain compatibility with the tensor structure (see [KL]IV, A.11).

A *balance* on $\mathcal{C}$ is an automorphism of the identity functor $b = \{b_M : M \xrightarrow{\sim} M \ (M \in \mathcal{C})\}$ such that for every $M, N \in \mathcal{C}$, $b_{M\otimes N} \circ (b_M \otimes b_N)^{-1} = R_{N,M} \circ R_{M,N}$ (see *loc. cit.*).

2.19. Let δ denote the determinant of the $I \times I$-matrix $(\langle i, j' \rangle)$. From now on we assume that $\mathrm{char}(k)$ does not divide 2δ, and k contains an element ζ' such that $(\zeta')^{2\delta} = \zeta$; we fix such an element ζ'. For a number $q \in \frac{1}{2\delta}\mathbb{Z}$, ζ^q will denote $(\zeta')^{2\delta q}$.

2.20. **Theorem.** (G. Lusztig) *There exists a unique braided structure $\{R_{M,N}\}$ on the tensor category $\mathcal{C}$ such that for any $\lambda \in X$ and $M \in \mathcal{C}$, if $\mu \in X$ is such that $M_{\mu'} \neq 0$ implies $\mu' \leq \mu$, then*

$$R_{L(\lambda),M}(x \otimes y) = \zeta^{\lambda\cdot\mu} y \otimes x$$

for any $x \in L(\lambda), y \in M_\mu$. $\square$

2.21. Let $n : X \longrightarrow \frac{1}{2\delta}\mathbb{Z}$ be the function defined by

$$n(\lambda) = \frac{1}{2}\lambda\cdot\lambda - \lambda\cdot\rho_\ell.$$

2.22. **Theorem.** *There exists a unique balance b on $\mathcal{C}$ such that for any $\lambda \in X$, $b_{L(\lambda)} = \zeta^{n(\lambda)}$.* $\square$

2.23. The rigid tensor category $\mathcal{C}$, together with the additional structure given by the above braiding and balance, is a *ribbon category* in the sense of Turaev, cf. [K] and references therein.

3. Braiding local systems

3.1. For a topological space T and a finite set J, T^J will denote the space of all maps $J \longrightarrow T$ (with the topology of the cartesian product). Its points are J-tuples (t_j) of points of T. We denote by T^{Jo} the subspace consisting of all (t_j) such that for any $j' \neq j''$ in J, $t_{j'} \neq t_{j''}$.

Let $\nu = \sum \nu_i i \in Y^+$. Let us call an *unfolding* of ν a map of finite sets $\pi : J \longrightarrow I$ such that $\mathrm{card}(\pi^{-1}(i)) = \nu_i$ for all i. Let Σ_π denote the group of all automorphisms $\sigma : J \xrightarrow{\sim} J$ such that $\pi \circ \sigma = \pi$.

The group Σ_π acts on the space T^J in the obvious way. We denote by T^ν the quotient space T^J/Σ_ν. This space does not depend, up to a unique isomorphism, on the choice of an unfolding π. The points of T^ν are collections (t_j) of I-colored points of T, such that for any I, there are ν_i points of color i. We have the canonical projection $T^J \longrightarrow T^\nu$, also

to be denoted by π. We set $T^{\nu o} = \pi(T^{Jo})$. The map π restricted to T^{Jo} is an unramified Galois covering with the Galois group Σ_π.

3.2. For a real $r > 0$, let $D(r)$ denote the open disk on the complex plane $\{t \in \mathbb{C}|\ |t| < r\}$ and $\bar{D}(r)$ its closure. For $r_1 < r_2$, denote by $A(r_1, r_2)$ the open annulus $D(r_2) - \bar{D}(r_1)$.

Set $D = D(1)$. Let A denote the punctured disk $D - \{0\}$.

3.3. For an integer $n \geq 1$, consider the space

$$E_n = \{(r_0, \ldots, r_n) \in \mathbb{R}^{n+1}|\ 0 = r_0 < r_1 < \ldots < r_n = 1\}.$$

Obviously, the space E_n is contractible.

Let $J_1, \ldots, J_n$ be finite sets. Set $J = \coprod_{a=1}^n J_a$. Note that $D^J = D^{J_1} \times \ldots \times D^{J_n}$. Let us untroduce the subspace $D^{J_1, \cdots, J_n} \subset E_n \times D^J$. By definition it consists of points $((r_a); (t^a_j) \in D^{J_a},\ a = 1, \ldots, n)$ such that $t^a_j \in A(r_{a-1}, r_a)$ for $a = 1, \ldots, n$. The canonical projection $E_n \times D^J \longrightarrow D^J$ induces the map

$$m(J_1, \ldots, J_n) : D^{J_1, \cdots, J_n} \longrightarrow D^J.$$

The image of the above projection lands in the subspace $D^{J_1} \times A^{J_2} \times \ldots \times A^{J_n} \subset D^{J_1} \times \ldots \times D^{J_n} = D^J$. The induced map

$$p(J_1, \ldots, J_n) : D^{J_1, \cdots, J_n} \longrightarrow D^{J_1} \times A^{J_2} \times \ldots \times A^{J_n}$$

is homotopy equivalence.

Now assume that we have maps $\pi_a : J_a \longrightarrow I$ which are unfoldings of the elements ν_a. Then their sum $\pi : J \longrightarrow I$ is an unfolding of $\nu = \nu_1 + \ldots + \nu_n$. We define the space $D^{\nu_1, \cdots, \nu_n} \subset E_n \times D^\nu$ as the image of $D^{J_1, \cdots, J_n}$ under the projection $\mathrm{Id} \times \pi : E_n \times D^J \longrightarrow E_n \times D^\nu$. The maps $m(J_1, \ldots, J_n)$ and $p(J_1, \ldots, J_n)$ induce the maps

$$m(\nu_1, \ldots, \nu_n) : D^{\nu_1, \cdots, \nu_n} \longrightarrow D^{\nu_1 + \ldots + \nu_n}$$

and

$$p(\nu_1, \ldots, \nu_n) : D^{\nu_1, \cdots, \nu_n} \longrightarrow D^{\nu_1} \times A^{\nu_2} \times \ldots \times A^{\nu_n}$$

respectively, the last map being homotopy equivalence.

3.4. We define the open subspaces

$$A^{\nu_1, \cdots, \nu_n} = D^{\nu_1, \cdots, \nu_n} \cap (E_n \times A^\nu)$$

and

$$A^{\nu_1, \cdots, \nu_n o} = D^{\nu_1, \cdots, \nu_n} \cap (E_n \times A^{\nu o}).$$

We have the maps

$$m_a(\nu_1, \ldots, \nu_n) : D^{\nu_1, \cdots, \nu_n} \longrightarrow D^{\nu_1, \cdots, \nu_{a-1}, \nu_a + \nu_{a+1}, \nu_{a+2}, \cdots, \nu_n}$$

and

$$p_a(\nu_1, \ldots, \nu_n) : D^{\nu_1, \cdots, \nu_n} \longrightarrow D^{\nu_1, \cdots, \nu_a} \times A^{\nu_{a+1}, \cdots, \nu_n}$$

($a = 1, \ldots, n-1$) defined in an obvious manner. They induce the maps

$$m_a(\nu_1, \ldots, \nu_n) : A^{\nu_1, \cdots, \nu_n} \longrightarrow A^{\nu_1, \cdots, \nu_{a-1}, \nu_a + \nu_{a+1}, \nu_{a+2}, \cdots, \nu_n}$$

and

$$p_a(\nu_1, \ldots, \nu_n) : A^{\nu_1, \cdots, \nu_n} \longrightarrow A^{\nu_1, \cdots, \nu_a} \times A^{\nu_{a+1}, \cdots, \nu_n}$$

and similar maps between "o"-ed spaces. All the maps p are homotopy equivalences.

All these maps satisfy some obvious compatibilities. We will need the following particular case.

3.5. The *rhomb* diagram below commutes.

$$\begin{tikzcd}
 & & D^{\nu_1+\nu_2+\nu_3} & & \\
 & D^{\nu_1,\nu_2+\nu_3} \arrow[ur, "m"] \arrow[dl, "p"'] & & D^{\nu_1+\nu_2,\nu_3} \arrow[ul, "m"'] \arrow[dr, "p"] & \\
D^{\nu_1} \times A^{\nu_1+\nu_2} & & D^{\nu_1,\nu_2,\nu_3} \arrow[ul, "m"] \arrow[ur, "m"'] \arrow[dl, "p"'] \arrow[dr, "p"] & & D^{\nu_1+\nu_2} \times A^{\nu_3} \\
 & D^{\nu_1} \times A^{\nu_1,\nu_2} \arrow[ul, "m"] \arrow[dr, "p"'] & & D^{\nu_1,\nu_2} \times A^{\nu_3} \arrow[ur, "m"'] \arrow[dl, "p"] & \\
 & & D^{\nu_1} \times A^{\nu_2} \times A^{\nu_3} & &
\end{tikzcd}$$

3.6. We will denote by $\mathcal{A}^o$ and call an *open I-coloured configuration space* the collection of all spaces $\{A^{\nu_1, \cdots, \nu_n o}\}$ together with the maps $\{m_a(\nu_1, \ldots, \nu_n), p_a(\nu_1, \ldots, \nu_n)\}$ between their various products.

We will call a *local system* over $\mathcal{A}^o$, or a *braiding local system* a collection of data (a), (b) below satisfying the property (c) below.

(a) A local system $\mathcal{I}_\mu^\nu$ over $A^{\nu o}$ given for any $\nu \in Y^+, \mu \in X$.

(b) An isomorphism $\phi_\mu(\nu_1, \nu_2) : m^* \mathcal{I}_\mu^{\nu_1+\nu_2} \xrightarrow{\sim} p^*(\mathcal{I}_\mu^{\nu_1} \boxtimes \mathcal{I}_{\mu-\nu_1}^{\nu_2})$ given for any $\nu_1, \nu_2 \in Y^+, \mu \in X$.

Here $p = p(\nu_1, \nu_2)$ and $m = m(\nu_1, \nu_2)$ are the arrows in the diagram $A^{\nu_1 o} \times A^{\nu_2 o} \xleftarrow{p} A^{\nu_1, \nu_2 o} \xrightarrow{m} A^{\nu_1+\nu_2 o}$.

The isomorphisms $\phi_\mu(\nu_1, \nu_2)$ are called the *factorization isomorphisms.*

(c) (The *associativity* of factorization isomorphisms.) For any $\nu_1, \nu_2, \nu_3 \in Y^+, \mu \in X$, the octagon below commutes. Here the maps m, p are the maps in the rhombic diagram above, with D, A replaced by A^o.

$$\begin{tikzcd}
 & m^*m_1^*\mathcal{I}_\mu^{\nu_1+\nu_2+\nu_3} \arrow[r, equal] \arrow[dl, "\phi_\mu(\nu_1,\nu_2+\nu_3)"'] & m^*m_2^*\mathcal{I}_\mu^{\nu_1+\nu_2+\nu_3} \arrow[dr, "\phi_\mu(\nu_1+\nu_2,\nu_3)"] & \\
m^*p^*(\mathcal{I}_\mu^{\nu_1}\boxtimes\mathcal{I}_{\mu-\nu_1}^{\nu_2+\nu_3}) \arrow[d, equal] & & & m^*p^*(\mathcal{I}_\mu^{\nu_1+\nu_2}\boxtimes\mathcal{I}_{\mu-\nu_1-\nu_2}^{\nu_3}) \arrow[d, equal] \\
p^*m^*(\mathcal{I}_\mu^{\nu_1}\boxtimes\mathcal{I}_{\mu-\nu_1}^{\nu_2+\nu_3}) \arrow[dr, "\phi_{\mu-\nu_1}(\nu_2,\nu_3)"'] & & & p^*m^*(\mathcal{I}_\mu^{\nu_1+\nu_2}\boxtimes\mathcal{I}_{\mu-\nu_1-\nu_2}^{\nu_3}) \arrow[dl, "\phi_\mu(\nu_1,\nu_2)"] \\
 & p_1^*p^*(\mathcal{I}_\mu^{\nu_1}\boxtimes\mathcal{I}_{\mu-\nu_1}^{\nu_2}\boxtimes\mathcal{I}_{\mu-\nu_1-\nu_2}^{\nu_3}) \arrow[r, equal] & p_2^*p^*(\mathcal{I}_\mu^{\nu_1}\boxtimes\mathcal{I}_{\mu-\nu_1}^{\nu_2}\boxtimes\mathcal{I}_{\mu-\nu_1-\nu_2}^{\nu_3}) &
\end{tikzcd}$$

Written more concisely, the axiom (c) reads as a "cocycle" condition

(c)$'$ $\phi_\mu(\nu_1,\nu_2)\circ\phi_\mu(\nu_1+\nu_2,\nu_3)=\phi_{\mu-\nu_1}(\nu_2,\nu_3)\circ\phi_\mu(\nu_1,\nu_2+\nu_3)$.

3.7. **Correctional lemma.** *Assume we are given the data* (a), (b) *as above, with one-dimensional local systems $\mathcal{I}_\mu^\nu$. Then there exist a collection of constants $c_\mu(\nu_1,\nu_2)\in k^*$ $(\mu\in X,\nu_1,\nu_2\in Y^+)$ such that the corrected isomorphisms $\phi'_\mu(\nu_1,\nu_2)=c_\mu(\nu_1,\nu_2)\phi_\mu(\nu_1,\nu_2)$ satisfy the associativity axiom* (c). $\square$

3.8. The notion of a morphism between two braiding local systems is defined in an obvious way. This defines the category of braiding local systems, $\mathcal{B}ls$.

Suppose that $\mathcal{I}=\{\mathcal{I}_\mu^\nu;\phi_\mu(\nu_1,\nu_2)\}$ and $\mathcal{J}=\{\mathcal{J}_\mu^\nu;\psi_\mu(\nu_1,\nu_2)\}$ are two braiding local systems. Their *tensor product* $\mathcal{I}\otimes\mathcal{J}$ is defined by $(\mathcal{I}\otimes\mathcal{J})_\mu^\nu=\mathcal{I}_\mu^\nu\otimes\mathcal{J}_\mu^\nu$, the factorization isomorphisms being $\phi_\mu(\nu_1,\nu_2)\otimes\psi_\mu(\nu_1,\nu_2)$. This makes $\mathcal{B}ls$ a tensor category. The subcategory of one-dimensional braiding local systems is a Picard category.

3.9. **Example.** *Sign local system.* Let L be a one-dimensional vector space. Let $\nu\in Y^+$. Let $\pi:J\longrightarrow I$ be an unfolding of ν, whence the canonical projection $\pi:A^{J\circ}\longrightarrow A^{\nu\circ}$. Pick a point $\tilde{x}=(x_j)\in A^{J\circ}$ with all x_j being *real*; let $x=\pi(\tilde{x})$. The choice of base points defines the homomorphism $\pi_1(A^{\nu\circ};x)\longrightarrow\Sigma_\pi$. Consider its composition with the sign map $\Sigma_\pi\longrightarrow\mu_2\hookrightarrow k^*$. Here $\mu_2=\{\pm1\}$ is the group of square roots of 1 in k^*. We get a map $s:\pi_1(A^{\nu\circ};x)\longrightarrow k^*$. It defines a local system $\mathcal{S}ign^\nu(L)$ over $A^{\nu\circ}$ whose stalk

at x is equal to L. This local system does not depend (up to the unique isomorphism) on the choices made.

Given a diagram

$$A^{\nu_1\circ}\times A^{\nu_2\circ}\xleftarrow{p}A^{\nu_1,\nu_2\circ}\xrightarrow{m}A^{\nu_2+\nu_2\circ}$$

we can choose base point $x_i\in A^{\nu_i}$ and $x\in A^{\nu_1+\nu_2\circ}$ such that there exists $y\in A^{\nu_1,\nu_2}$ with $p(y)=(x_1,x_2)$ and $m(y)=x$. Let

$$\phi^{L_1,L_2}(\nu_1,\nu_2):m^*\mathcal{S}ign^{\nu_1+\nu_2}(L_1\otimes L_2)\overset{\sim}{\longrightarrow}p^*(\mathcal{S}ign^{\nu_1}(L_1)\boxtimes\mathcal{S}ign^{\nu_2}(L_2))$$

be the unique isomorphism equal to $\mathrm{Id}_{L_1\otimes L_2}$ over y. These isomorphisms satisfy the associativity condition (cf. (c)$'$ above).

Let us define the brading local system $\mathcal{S}ign$ by $\mathcal{S}ign^\nu_\mu=\mathcal{S}ign^\nu(\mathbf{1})$, $\phi_\mu(\nu_1,\nu_2)=\phi^{\mathbf{1},\mathbf{1}}(\nu_1,\nu_2)$. Here $\mathbf{1}$ denotes the standard vector space k.

3.10. **Example.** *Standard local systems $\mathcal{J}$, $\mathcal{I}$.* Let $\nu\in Y^+$, let $\pi:J\longrightarrow I$ be an unfolding of ν, $n=\mathrm{card}(J)$, $\pi:A^{J\circ}\longrightarrow A^{\nu\circ}$ the corresponding projection. Let L be a one-dimensional vector space. For each isomorphism $\sigma:J\overset{\sim}{\longrightarrow}\{1,\dots,n\}$, choose a point $x(\sigma)=(x_j)\in A^{J\circ}$ such that all x_j are real and positive and for each j',j'' such that $\sigma(j')<\sigma(j'')$, we have $x_{j'}<x_{j''}$.

Let us define a local system $\mathcal{J}^\pi_\mu(L)$ over $A^{J\circ}$ as follows. We set $\mathcal{J}^\pi_\mu(L)_{x(\sigma)}=L$ for all σ. We define the monodromies

$$T_\gamma:\mathcal{J}^\pi_\mu(L)_{x(\sigma)}\longrightarrow\mathcal{J}^\pi_\mu(L)_{x(\sigma')}$$

along the homotopy classes of paths $\gamma:x_\sigma\longrightarrow x_{\sigma'}$ which generate the fundamental groupoid. Namely, let $x_{j'}$ and $x_{j''}$ be some neighbour points in $x(\sigma)$, with $x_{j'}<x_{j''}$. Let $\gamma(j',j'')^+$ (resp. $\gamma(j',j'')^-$) be the paths corresponding to the movement of $x_{j''}$ in the upper (resp. lower) hyperplane to the left from $x_{j'}$ position. We set

$$T_{\gamma(j',j'')^\pm}=\zeta^{\mp\pi(j')\cdot\pi(j'')}.$$

Let x_j be a point in $x(\sigma)$ closest to 0. Let $\gamma(j)$ be the path corresponding to the counterclockwise travel of x_j around 0. We set

$$T_{\gamma(j)}=\zeta^{2\mu\cdot\pi(j)}.$$

The point is that the above fromulas give a well defined morphism from the fundamental groupoid to the groupoid of one-dimensional vector spaces.

The local system $\mathcal{J}^\pi_\mu(L)$ admits an obvious Σ_π-equivariant structure. This defines the local system $\mathcal{J}^\nu_\mu(L)$ over $A^{\nu\circ}$. Given a diagram

$$A^{\nu_1\circ}\times A^{\nu_2\circ}\xleftarrow{p}A^{\nu_1+\nu_2\circ}\xrightarrow{m}A^{\nu_1+\nu_2\circ},$$

let

$$\phi^{L_1,L_2}_\mu(\nu_1,\nu_2):m^*\mathcal{J}^{\nu_1+\nu_2}_\mu(L_1\otimes L_2)\overset{\sim}{\longrightarrow}p^*(\mathcal{J}^{\nu_1}_\mu(L_1)\boxtimes\mathcal{J}^{\nu_2}_{\mu-\nu_1}(L_2))$$

be the unique isomorphism equal to $Id_{L_1\otimes L_2}$ over compatible base points. Here the compatibility is understood in the same sense as in the previous subsection. These isomorphisms satisfy the associativity condition.

We define the braiding local system $\mathcal{J}$ by $\mathcal{J}^\nu_\mu = \mathcal{J}^\nu_\mu(\mathbf{1})$, $\phi_\mu(\nu_1,\nu_2) = \phi_\mu^{\mathbf{1},\mathbf{1}}(\nu_1,\nu_2)$.

We define the braiding local system $\mathcal{I}$ by $\mathcal{I} = \mathcal{J} \otimes \mathcal{S}ign$. The local systems $\mathcal{J}$, $\mathcal{I}$ are called the standard local systems. In the sequel we will mostly need the local system $\mathcal{I}$.

4. Factorizable sheaves

4.1. In the sequel for each $\nu \in Y^+$, we will denote by $\mathcal{S}$ the stratification on the space D^ν the closures of whose strata are various intersections of hypersurfaces given by the equations $t_j = 0,\ t_{j'} = t_{j''}$. The same letter will denote the induced stratifications on its subspaces.

4.2. Set $\mathcal{I}^{\nu\bullet}_\mu = j_{!*}\mathcal{I}^\nu_\mu[\dim A^{\nu\circ}] \in \mathcal{M}(A^\nu;\mathcal{S})$ where $j : A^{\nu\circ} \hookrightarrow A^\nu$ is the embedding.

Given a diagram

$$A^{\nu_1}\times A^{\nu_2} \xleftarrow{p} A^{\nu_1,\nu_2} \xrightarrow{m} A^{\nu_1+\nu_2},$$

the structure isomorphisms $\phi_\mu(\nu_1,\nu_2)$ of the local system $\mathcal{I}$ induce the isomorphisms, to be denoted by the same letters,

$$\phi_\mu(\nu_1,\nu_2) : m^*\mathcal{I}^{\nu_1+\nu_2\bullet} \xrightarrow{\sim} p^*(\mathcal{I}^{\nu_1\bullet}\boxtimes\mathcal{I}^{\nu_2\bullet}).$$

Obviously, these isomorphisms satisfy the associativity axiom.

4.3. Let us fix a coset $c \in X/Y$. We will regard c as a subset of X. We will call a *factorizable sheaf $\mathcal{M}$ supported at c* a collection of data (w), (a), (b) below satisfying the axiom (c) below.

(w) An element $\lambda = \lambda(\mathcal{M}) \in c$.

(a) A perverse sheaf $\mathcal{M}^\nu \in \mathcal{M}(D^\nu;\mathcal{S})$ given for each $\nu \in Y^+$.

(b) An isomorphism $\psi(\nu_1,\nu_2) : m^*\mathcal{M}^{\nu_1+\nu_2} \xrightarrow{\sim} p^*(\mathcal{M}^{\nu_1}\boxtimes\mathcal{I}^{\nu_2\bullet}_{\lambda-\nu_1})$ given for any $\nu_1,\nu_2 \in Y^+$.

Here p, m denote the arrows in the diagram $D^{\nu_1}\times A^{\nu_2} \xleftarrow{p} D^{\nu_1,\nu_2} \xrightarrow{m} D^{\nu_1+\nu_2}$.

The isomorphisms $\psi(\nu_1,\nu_2)$ are called the *factorization isomorphisms*.

(c) For any $\nu_1,\nu_2,\nu_3 \in Y^+$, the following *associativity condition* is fulfilled:

$$\psi(\nu_1,\nu_2)\circ\psi(\nu_1+\nu_2,\nu_3) = \phi_{\lambda-\nu_1}(\nu_2,\nu_3)\circ\psi(\nu_1,\nu_2+\nu_3).$$

We leave to the reader to draw the whole octagon expressing this axiom.

4.4. Let $\mathcal{M}=\{\mathcal{M}^\nu;\ \psi(\nu_1,\nu_2)\}$ be a factorizable sheaf supported at a coset $c\in X/Y$, $\lambda=\lambda(\mathcal{M})$. For each $\lambda'\geq\lambda,\nu\in Y^+$, define a sheaf $\mathcal{M}^\nu_{\lambda'}\in\mathcal{M}(D^\nu;\mathcal{S})$ by

$$\mathcal{M}^\nu_{\lambda'}=\begin{cases}\iota(\lambda'-\lambda)_*\mathcal{M}^{\nu-\lambda'+\lambda} & \text{if } \nu-\lambda'+\lambda\in Y^+\\ 0 & \text{otherwise.}\end{cases}$$

Here

$$\iota(\nu'):D^\nu\longrightarrow D^{\nu+\nu'}$$

denotes the closed embedding adding ν' points sitting at the origin. The factorization isomorphisms $\psi(\nu_1,\nu_2)$ induce similar isomorphisms

$$\psi_{\lambda'}(\nu_1,\nu_2):m^*\mathcal{M}_{\lambda'}(\nu_1+\nu_2)\xrightarrow{\sim}p^*(\mathcal{M}^{\nu_1}_{\lambda'}\boxtimes\mathcal{I}^{\nu_2}_{\lambda'-\nu_1})\ (\lambda'\geq\lambda)$$

4.5. Let $\mathcal{M},\mathcal{N}$ be two factorizable sheaves supported at c. Let $\lambda\in X$ be such that $\lambda\geq\lambda(\mathcal{M})$ and $\lambda\geq\lambda(\mathcal{N})$. For $\nu\geq\nu'$ in Y^+, consider the following composition

$$\tau_\lambda(\nu,\nu'):\mathrm{Hom}(\mathcal{M}^\nu_\lambda,\mathcal{N}^\nu_\lambda)\xrightarrow{m_*}\mathrm{Hom}(m^*\mathcal{M}^\nu_\lambda,m^*\mathcal{N}^\nu_\lambda)\overset{\psi(\nu',\nu-\nu')_*}{\xrightarrow{\sim}}$$

$$\xrightarrow{\sim}\mathrm{Hom}(p^*(\mathcal{M}^{\nu'}_\lambda\boxtimes\mathcal{I}^{\nu-\nu'\bullet}_{\lambda-\nu'}),p^*(\mathcal{N}^{\nu'}_\lambda\boxtimes\mathcal{I}^{\nu-\nu'\bullet}_{\lambda-\nu'}))=\mathrm{Hom}(\mathcal{M}^{\nu'}_\lambda,\mathcal{N}^{\nu'}_\lambda).$$

Let us define the space of homomorphisms $\mathrm{Hom}(\mathcal{M},\mathcal{N})$ by

$$\mathrm{Hom}(\mathcal{M},\mathcal{N})=\lim_{\to\ \lambda}\lim_{\leftarrow\ \nu}\mathrm{Hom}(\mathcal{M}^\nu_\lambda,\mathcal{N}^\nu_\lambda)$$

Here the inverse limit is taken over $\nu\in Y^+$, the transition maps being $\tau_\lambda(\nu,\nu')$ and the direct limit is taken over $\lambda\in X$ such that $\lambda\geq\lambda(\mathcal{M}),\lambda\geq\lambda(\mathcal{N})$, the transition maps being induced by the obvious isomorphisms

$$\mathrm{Hom}(\mathcal{M}^\nu_\lambda,\mathcal{N}^\nu_\lambda)=\mathrm{Hom}(\mathcal{M}^{\nu+\nu'}_{\lambda+\nu'},\mathcal{N}^{\nu+\nu'}_{\lambda+\nu'})\ (\nu'\in Y^+).$$

With these spaces of homomorphisms and the obvious compositions, the factorizable sheaves supported at c form the category, to be denoted by $\widetilde{\mathcal{FS}}_c$. By definition, the category $\widetilde{\mathcal{FS}}$ is the direct product $\prod_{c\in X/Y}\ \widetilde{\mathcal{FS}}_c$.

4.6. **Finite sheaves.** Let us call a factorizable sheaf $\mathcal{M}=\{\mathcal{M}^\nu\}\in\widetilde{\mathcal{FS}}_c$ *finite* if there exists only a finite number of $\nu\in Y^+$ such that the conormal bundle of the origin $O\in\mathcal{A}^\nu$ is contained in the singular support of $\mathcal{M}^\nu$. Let $\mathcal{FS}_c\subset\widetilde{\mathcal{FS}}_c$ be the full subcategory of finite factorizable sheaves. We define the category $\mathcal{FS}$ by $\mathcal{FS}=\prod_{c\in X/Y}\mathcal{FS}_c$. One proves (using the lemma below) that $\mathcal{FS}$ is an abelian category.

4.7. **Stabilization Lemma.** *Let $\mathcal{M},\mathcal{N}\in\mathcal{FS}_c,\ \mu\in X_c,\mu\geq\lambda(\mathcal{M}),\mu\geq\lambda(\mathcal{N})$. There exists $\nu_0\in Y^+$ such that for all $\nu\geq\nu_0$ the canonical maps*

$$\mathrm{Hom}(\mathcal{M},\mathcal{N})\longrightarrow\mathrm{Hom}(\mathcal{M}^\nu_\mu,\mathcal{N}^\nu_\mu)$$

are isomorphisms. $\square$

4.8. **Standard sheaves.** Given $\mu \in X$, let us define the "standard sheaves" $\mathcal{M}(\mu), \mathcal{DM}(\mu)$ and $\mathcal{L}(\mu)$ supported at the coset $\mu + Y$, by $\lambda(\mathcal{M}(\mu)) = \lambda(\mathcal{DM}(\mu)) = \lambda(\mathcal{L}(\mu)) = \mu$;

$$\mathcal{M}(\mu)^\nu = j_! \mathcal{I}_\mu^{\nu\bullet};\ \mathcal{DM}(\mu)^\nu = j_* \mathcal{I}_\mu^{\nu\bullet};\ \mathcal{L}(\mu)^\nu = j_{!*} \mathcal{I}_\mu^{\nu\bullet},$$

j being the embedding $A^\nu \hookrightarrow D^\nu$. The factorization maps are defined by functoriality from the similar maps for $\mathcal{I}^\bullet$.

One proves that all these sheaves are finite.

5. Tensor product

In this section we will give (a sketch of) the construction of the tensor structure on the category $\widetilde{\mathcal{FS}}$. We will make the assumption of 2.19[5].

5.1. For $z \in \mathbb{C}$ and a real positive r, let $D(z; r)$ denote the open disk $\{t \in \mathbb{C}|\ |t - z| < r\}$ and $\bar{D}(z; r)$ its closure.

5.2. For $\nu \in Y^+$, let us define the space $D^\nu(2)$ as the product $A \times D^\nu$. Its points will be denoted $(z; (t_j))$ where $z \in A, (t_j) \in D^\nu$. Let us define the open subspaces

$$A^\nu(2) = \{(z; (t_j)) \in D^\nu(2)|\ t_j \neq 0, z \text{ for all } j\};\ A^\nu(2)^\circ = A^\nu(2) \cap (A \times A^{\nu\circ}).$$

For $\nu, \nu' \in Y^+$, let us define the space $D^{\nu,\nu'}(2)$ as the subspace of $\mathbb{R}_{>0} \times D^{\nu+\nu'}(2)$ consisting of all elements $(r; z; (t_j))$ such that $|z| < r < 1$ and ν of the points t_j live inside the disk $D(r)$ and ν' of them inside the annulus $A(r, 1)$.

We have a diagram

(a) $D^\nu(2) \times A^{\nu'} \xleftarrow{p} D^{\nu,\nu'}(2) \xrightarrow{m} D^{\nu+\nu'}(2)$.

Here $p((r; z; (t_j))) = ((z; (t_{j'})), (t_{j''}))$ where $t_{j'}$ (resp. $t_{j''}$) being the points from the collection (t_j) lying in $D(r)$ (resp. in $A(r, 1)$); $m((r; z; (t_j))) = (z; (t_j))$. The map p is a homotopy equivalence.

For $\nu_1, \nu_2, \nu \in Y^+$, let $D^{\nu_1;\nu_2;\nu}(2)$ be the subspace of $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times D^{\nu_1+\nu_2+\nu}(2)$ consisting of all elements $(r_1; r_2; z; (t_j))$ such that $\bar{D}(r_1) \cup \bar{D}(z; r_2) \subset D$; $\bar{D}(r_1) \cap \bar{D}(z; r_2) = \emptyset$; ν_1 of the points (t_j) lie inside $D(r_1)$, ν_2 of them lie inside $D(z; r_2)$ and ν of them lie inside $D - (\bar{D}(r_1) \cup \bar{D}(z; r_2))$.

We have a diagram

(b) $D^{\nu_1} \times D^{\nu_2} \times A^\nu(2) \xleftarrow{p} D^{\nu_1;\nu_2;\nu}(2) \xrightarrow{m} D^{\nu_1+\nu_2+\nu}(2)$.

Here $p((r_1; r_2; z; (t_j))) = ((t_{j'}); (t_{j''} - z); (t_{j'''}))$ where $t_{j'}$ (resp. $t_{j''}, t_{j'''}$) are the points lying inside $D(r_1)$ (resp. $D(z; r_2), D - (\bar{D}(r_1) \cup \bar{D}(z; r_2))$); $m((r_1; r_2; z; (t_j))) = (z; (t_j))$. The map p is a homotopy equivalence.

[5]Note that these assumptions are not necessary for the construction of the tensor structure. They are essential, however, for the construction of braiding.

5.3. We set $A^{\nu,\nu'}(2) = D^{\nu,\nu'}(2)\cap(\mathbb{R}_{>0}\times A^{\nu+\nu'}(2))$; $A^{\nu,\nu'}(2)^\circ = D^{\nu,\nu'}(2)\cap(\mathbb{R}_{>0}\times A^{\nu+\nu'}(2)^\circ)$; $A^{\nu_1;\nu_2;\nu}(2) = D^{\nu_1;\nu_2;\nu}(2)\cap(\mathbb{R}_{>0}\times\mathbb{R}_{>0}\times A^{\mu_1+\nu_2+\nu}(2))$; $A^{\nu_1;\nu_2;\nu}(2)^\circ = D^{\nu_1;\nu_2;\nu}(2)\cap(\mathbb{R}_{>0}\times\mathbb{R}_{>0}\times A^{\nu_1+\nu_2+\nu}(2)^\circ)$.

5.4. Given $\mu_1,\mu_2\in X, \nu\in Y^+$, choose an unfolding $\pi : J\longrightarrow I$ of the element ν. In the same manner as in 3.10, we define the one-dimensional local system $\mathcal{J}^\nu_{\mu_1,\mu_2}$ over $A^\nu(2)^\circ$ with the following monodromies: the monodromy around a loop corresponding to the counterclockwise travel of the point z around 0 (resp. t_j around 0, t_j around z, $t_{j'}$ around $t_{j''}$) is equal to the multiplication by $\zeta^{-2\mu_1\cdot\mu_2}$ (resp. $\zeta^{2\mu_1\cdot\pi(j)}$, $\zeta^{2\mu_2\cdot\pi(j)}$, $\zeta^{-2\pi(j')\cdot\pi(j'')}$).

As in *loc. cit.*, one defines isomorphisms

(a) $\phi_{\mu_1,\mu_2}(\nu,\nu') : m^*\mathcal{J}^{\nu+\nu'}_{\mu_1,\mu_2}\stackrel{\sim}{\longrightarrow} p^*(\mathcal{J}^\nu_{\mu_1,\mu_2}\boxtimes\mathcal{J}^{\nu'}_{\mu_1+\mu_2-\nu})$

where p, m are the morphisms in the diagram 5.2 (a) (restricted to the A°-spaces) and

(b) $\phi_{\mu_1,\mu_2}(\nu_1;\nu_2;\nu) : m^*\mathcal{J}^{\nu_1+\nu_2+\nu}_{\mu_1,\mu_2}\stackrel{\sim}{\longrightarrow} p^*(\mathcal{J}^{\nu_1}_{\mu_1}\boxtimes\mathcal{J}^{\nu_2}_{\mu_2}\boxtimes\mathcal{J}^{\nu}_{\mu_1-\nu_1,\mu_2-\nu_2})$

where p,m are the morphisms in the diagram 5.2 (b) (restricted to the A°-spaces), which satisfy the cocycle conditions

(c) $\phi_{\mu_1,\mu_2}(\nu,\nu')\circ\phi_{\mu_1,\mu_2}(\nu+\nu',\nu'') = \phi_{\mu_1+\mu_2-\nu}(\nu',\nu'')\circ\phi_{\mu_1,\mu_2}(\nu,\nu'+\nu'')$

(d) $(\phi_{\mu_1}(\nu_1,\nu_1')\boxtimes\phi_{\mu_2}(\nu_2,\nu_2'))\circ\phi_{\mu_1,\mu_2}(\nu_1+\nu_1';\nu_2+\nu_2';\nu) =$

$= \phi_{\mu_1-\nu_1,\mu_2-\nu_2}(\nu_1';\nu_2';\nu)\circ\phi_{\mu_1,\mu_2}(\nu_1;\nu_2;\nu+\nu_1'+\nu_2')$

(we leave to the reader the definition of the corresponding spaces).

5.5. Let us consider the sign local systems introduced in 3.9. We will keep the same notation $\mathcal{S}ign^\nu$ for the inverse image of the local system $\mathcal{S}ign^\nu$ under the forgetting of z map $A^\nu(2)^\circ\longrightarrow A^{\nu\circ}$. We have the factorization isomorphisms

(a) $\phi^{\mathcal{S}ign}(\nu,\nu') : m^*\mathcal{S}ign^{\nu+\nu'}\stackrel{\sim}{\longrightarrow} p^*(\mathcal{S}ign^\nu\boxtimes\mathcal{S}ign^{\nu'})$;

(b) $\phi^{\mathcal{S}ign}(\nu_1;\nu_2;\nu) : m^*\mathcal{S}ign^{\nu_1+\nu_2+\nu}\stackrel{\sim}{\longrightarrow} p^*(\mathcal{S}ign^{\nu_1}\boxtimes\mathcal{S}ign^{\nu_2}\boxtimes\mathcal{S}ign^{\nu})$

which satisfy the cocycle conditions similar to (c), (d) above.

5.6. We define the local systems $\mathcal{I}^\nu_{\mu_1,\mu_2}$ over the spaces $A^\nu(2)^\circ$ by

$\mathcal{I}^\nu_{\mu_1,\mu_2} = \mathcal{J}^\nu_{\mu_1,\mu_2}\otimes\mathcal{S}ign^\nu$.

The collection of local systems $\{\mathcal{I}^\nu_{\mu_1,\mu_2}\}$ together with the maps $\phi^{\mathcal{I}}_{\mu_1,\mu_2}(\nu,\nu') = \phi^{\mathcal{J}}_{\mu_1,\mu_2}(\nu,\nu')\otimes\phi^{\mathcal{S}ign}(\nu,\nu')$ and $\phi^{\mathcal{I}}_{\mu_1,\mu_2}(\nu_1;\nu_2;\nu) = \phi^{\mathcal{J}}_{\mu_1,\mu_2}(\nu_1;\nu_2;\nu)\otimes\phi^{\mathcal{S}ign}(\nu_1;\nu_2;\nu)$, forms an object $\mathcal{I}(2)$ which we call a *standard braiding local system over the configuration space* $\mathcal{A}(2)^\circ = \{A^\nu(2)^\circ\}$. It is unique up to a (non unique) isomorphism. We fix such a local system.

5.7. We set $\mathcal{I}^{\nu\bullet}_{\mu_1,\mu_2} = j_{!*}\mathcal{I}^\nu_{\mu_1,\mu_2}[\dim A^\nu(2)^\circ]$ where $j : A^\nu(2)^\circ\hookrightarrow A^\nu(2)$ is the open embedding. It is an object of the category $\mathcal{M}(A^\nu(2);\mathcal{S})$ where $\mathcal{S}$ is the evident stratification.

The factorization isomorphisms for the local system $\mathcal{I}$ induce the analogous isomorphisms between these sheaves, to be denoted by the same letter. The collection of these sheaves and factorization isomorphisms will be denoted $\mathcal{I}(2)^\bullet$.

5.8. Suppose we are given two factorizable sheaves $\mathcal{M}, \mathcal{N}$. Let us call their *gluing*, and denote by $\mathcal{M} \boxtimes \mathcal{N}$, the collection of perverse sheaves $(\mathcal{M} \boxtimes \mathcal{N})^\nu$ over the spaces $D^\nu(2)$ $(\nu \in Y^+)$ together with isomorphisms

$$\psi(\nu_1; \nu_2; \nu) : m^*(\mathcal{M} \boxtimes \mathcal{N})^\nu \xrightarrow{\sim} p^*(\mathcal{M}^{\nu_1} \boxtimes \mathcal{N}^{\nu_2} \boxtimes \mathcal{I}^{\nu\bullet}_{\lambda(\mathcal{M})-\nu_1, \lambda(\mathcal{N})-\nu_2}),$$

p, m being the maps in the diagram 5.2 (b), which satisfy the cocycle condition

$$(\psi^{\mathcal{M}}(\nu_1, \nu_1') \boxtimes \psi^{\mathcal{N}}(\nu_2, \nu_2')) \circ \psi(\nu_1 + \nu_1'; \nu_2 + \nu_2'; \nu) =$$
$$= \phi_{\lambda(\mathcal{M})-\nu_1, \lambda(\mathcal{N})-\nu_2}(\nu_1'; \nu_2'; \nu) \circ \psi(\nu_1; \nu_2; \nu + \nu_1' + \nu_2')$$

for all $\nu_1, \nu_1', \nu_2, \nu_2', \nu \in Y^+$.

Such a gluing exists and is unique, up to a unique isomorphism. The factorization isomorphisms $\phi_{\mu_1,\mu_2}(\nu_1; \nu_2; \nu)$ for $\mathcal{I}(2)^\bullet$ and the ones for $\mathcal{M}, \mathcal{N}$, induce the isomorphisms

$$\psi^{\mathcal{M}\boxtimes\mathcal{N}}(\nu, \nu') : m^*(\mathcal{M} \boxtimes \mathcal{N})^{\nu+\nu'} \xrightarrow{\sim} p^*((\mathcal{M} \boxtimes \mathcal{N})^\nu \boxtimes \mathcal{I}^{\nu'\bullet}_{\lambda(\mathcal{M})+\lambda(\mathcal{N})-\nu}).$$

satisfying the obvious cocycle condition.

5.9. Now we can define the tensor product $\mathcal{M} \otimes \mathcal{N} \in \widetilde{\mathcal{FS}}$. Namely, set $\lambda(\mathcal{M} \otimes \mathcal{N}) = \lambda(\mathcal{M}) + \lambda(\mathcal{N})$. For each $\nu \in Y^+$, set

$$(\mathcal{M} \otimes \mathcal{N})^\nu = \Psi_{z \to 0}((\mathcal{M} \boxtimes \mathcal{N})^\nu).$$

Here $\Psi_{z\to 0} : \mathcal{M}(D^\nu(2)) \longrightarrow \mathcal{M}(D^\nu)$ denotes the functor of nearby cycles for the function $D^\nu(2) \longrightarrow D$ sending $(z; (t_j))$ to z. Note that

$$\Psi_{z\to 0}(\mathcal{I}^\nu_{\mu_1,\mu_2}) = \mathcal{I}^\nu_{\mu_1+\mu_2}.$$

The factorization isomorphisms $\psi^{\mathcal{M}\boxtimes\mathcal{N}}$ induce the factorization isomorphisms between the sheaves $(\mathcal{M} \otimes \mathcal{N})^\nu$. This defines a factorizable sheaf $\mathcal{M} \otimes \mathcal{N}$.

One sees at once that this construction is functorial; thus it defines a functor of tensor product $\otimes : \widetilde{\mathcal{FS}} \times \widetilde{\mathcal{FS}} \longrightarrow \widetilde{\mathcal{FS}}$.

The subcategory $\mathcal{FS} \subset \widetilde{\mathcal{FS}}$ is stable under the tensor product. The functor $\otimes : \mathcal{FS} \times \mathcal{FS} \longrightarrow \mathcal{FS}$ extends uniquely to a functor $\otimes : \mathcal{FS} \otimes \mathcal{FS} \longrightarrow \mathcal{FS}$ (for the discussion of the tensor product of abelian categories, see [D2] 5).

5.10. The half-circle travel of the point z around 0 from 1 to -1 in the upper halfplane defines the braiding isomorphisms

$$R_{\mathcal{M},\mathcal{N}} : \mathcal{M} \otimes \mathcal{N} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{M}.$$

We will not describe here the precise definition of the associativity isomorphisms for the tensor product $\otimes$. We just mention that to define them one should introduce into the

game certain configuration spaces $D^\nu(3)$ whose (more or less obvious) definition we leave to the reader.

The unit of this tensor structure is the sheaf $\mathbf{1} = \mathcal{L}(0)$ (cf. 4.8).

Equipped with these complementary structures, the category $\widetilde{\mathcal{FS}}$ becomes a *braided tensor category*.

6. Vanishing cycles

GENERAL GEOMETRY

6.1. Let us fix a finite set J, and consider the space D^J. Inside this space, let us consider the subspaces $D^J_{\mathbb{R}} = D^J \cap \mathbb{R}^J$ and $D^{J+} = D^J \cap \mathbb{R}^J_{\geq 0}$.

Let $\mathcal{H}$ be the set (*arrangement*) of all real hyperplanes in $D^J_{\mathbb{R}}$ of the form $H_j: \ t_j = 0$ or $H_{j'j''}: \ t_{j'} = t_{j''}$. An *edge* L of the arrangement $\mathcal{H}$ is a subspace of $D^J_{\mathbb{R}}$ which is a non-empty intersection $\bigcap H$ of some hyperplanes from $\mathcal{H}$. We denote by L° the complement $L - \bigcup L'$, the union over all edges $L' \subset L$ of smaller dimension. A *facet* of $\mathcal{H}$ is a connected component F of some L°. We call a facet *positive* if it lies entirely inside D^{J+}.

For example, we have a unique smallest facet O — the origin. For each $j \in J$, we have a positive one-dimensional facet F_j given by the equations $t_{j'} = 0 \ (j' \neq j); \ t_j \geq 0$.

Let us choose a point w_F on each positive facet F. We call a *flag* a sequence of embedded positive facets $\mathbf{F}: \ F_0 \subset \ldots F_p$; we say that $\mathbf{F}$ *starts* from F_0. To such a flag we assign the simplex $\Delta_{\mathbf{F}}$ — the convex hull of the points $w_{F_0}, \ldots, w_{F_p}$.

To each positive facet F we assign the following two spaces: $D_F = \bigcup \Delta_{\mathbf{F}}$, the union over all flags $\mathbf{F}$ starting from F, and $S_F = \bigcup \Delta_{\mathbf{F}'}$, the union over all flags $\mathbf{F}'$ starting from a facet which properly contains F. Obviously, $S_F \subset D_F$.

6.2. Given a complex $\mathcal{K} \in \mathcal{D}(D^J; \mathcal{S})$ and a positive facet F, we introduce a complex of vector spaces $\Phi_F(\mathcal{K})$ by

$$\Phi_F(\mathcal{K}) = R\Gamma(D_F, S_F; \mathcal{K})[-\dim F].$$

This is a well defined object of the bounded derived category $\mathcal{D}(*)$ of finite dimensional vector spaces, not depending on the choice of points w_F. It is called the *complex of vanishing cycles of* $\mathcal{K}$ *across* F.

6.3. **Theorem.** *We have canonically* $\Phi_F(D\mathcal{K}) = D\Phi_F(\mathcal{K})$ *where* D *denotes the Verdier duality in the corresponding derived categories.* $\square$

6.4. **Theorem.** *If $\mathcal{M} \in \mathcal{M}(D^J;\mathcal{S})$ then $H^i(\Phi_F(\mathcal{M})) = 0$ for $i \neq 0$. Thus, Φ_F induces an exact functor*

$$\Phi_F : \mathcal{M}(D^J;\mathcal{S}) \longrightarrow \mathcal{V}ect$$

to the category $\mathcal{V}ect$ of finite dimensional vector spaces. □

6.5. Given a positive facet E and $\mathcal{K} \in \mathcal{D}(D^J,\mathcal{S})$, we have $S_E = \bigcup_{F\in\mathcal{F}^1(E)} D_F$, the union over the set $\mathcal{F}^1(E)$ of all positive facets $F \supset E$ with $\dim F = \dim F + 1$, and

$$R\Gamma(S_E, \bigcup_{F\in\mathcal{F}^1(E)} S_F;\mathcal{K}) = \oplus_{F\in\mathcal{F}^1(E)} R\Gamma(D_F, S_F;\mathcal{K}).$$

6.6. For two positive facets E and $F \in \mathcal{F}^1(E)$, and $\mathcal{K}(D^J;\mathcal{S})$, let us define the natural map

$$u = u_E^F(\mathcal{K}) : \Phi_F(\mathcal{K}) \longrightarrow \Phi_E(\mathcal{K})$$

called *canonical*, as the composition

$$R\Gamma(D_F, S_F;\mathcal{K})[-p] \longrightarrow R\Gamma(S_E, \bigcup_{F'\in\mathcal{F}^1(E)} S_{F'};\mathcal{K})[-p] \longrightarrow R\Gamma(S_E;\mathcal{K})[-p] \longrightarrow$$

$$\longrightarrow R\Gamma(D_E, S_E)[-p+1]$$

where $p = \dim F$, the first arrow being induced by the equality in 6.5, the last one being the coboundary map.

Define the natural map

$$v = v_F^E(\mathcal{K}) : \Phi_E(\mathcal{K}) \longrightarrow \Phi_F(\mathcal{K})$$

called *variation*, as the map dual to the composition

$$D\Phi_F(\mathcal{K}) = \Phi_F(D\mathcal{K}) \stackrel{u(D\mathcal{K})}{\longrightarrow} \Phi_E(D\mathcal{K}) = D\Phi_E(\mathcal{K}).$$

BACK TO FACTORIZABLE SHEAVES

6.7. Let $\nu \in Y$. We are going to give two equivalent definitions of an exact functor, called *vanishing cycles at the origin*

$$\Phi : \mathcal{M}(D^\nu;\mathcal{S}) \longrightarrow \mathcal{V}ect.$$

First definition. Let $f : D^\nu \longrightarrow D$ be the function $f((t_j)) = \sum t_j$. For an object $\mathcal{K} \in \mathcal{D}(D^\nu;\mathcal{S})$, the Deligne's complex of vanishing cycles $\Phi_f(\mathcal{K})$ (cf. [D3]) is concentrated at the origin of the hypersurface $f^{-1}(0)$. It is t-exact with respect to the middle t-structure. We set by definition, $\Phi(\mathcal{M}) = H^0(\Phi_f(\mathcal{M}))$ ($\mathcal{M} \in \mathcal{M}(D^\nu;\mathcal{S})$).

Second definition. Choose an unfolding of ν, $\pi : J \longrightarrow I$. Let us consider the canonical projection $\pi : D^J \longrightarrow D^\nu$. For $\mathcal{K} \in \mathcal{D}(D^\nu;\mathcal{S})$, the complex $\pi^*\mathcal{K}$ is well defined as an element of the Σ_π-equivariant derived category, hence $\Phi_O(\pi^*(\mathcal{K}))$ is a well defined object of the Σ_π-equivariant derived category of vector spaces (O being the origin facet in D^J). Therefore, the complex of Σ_π-invariants $\Phi_O(\pi^*\mathcal{K})^{\Sigma_\pi}$ is a well defined object of $\mathcal{D}(*)$. If

$\mathcal{K} \in \mathcal{M}(D^\nu;\mathcal{S})$ then all the cohomology of $\Phi_O(\pi^*\mathcal{K})^{\Sigma_\pi}$ in non-zero degree vanishes. We set

$$\Phi(\mathcal{M}) = H^0(\Phi_O(\pi^*\mathcal{M})^{\Sigma_\pi})$$

$(\mathcal{M} \in \mathcal{M}(D^\nu;\mathcal{S}))$. The equivalence of the two definitions follows without difficulty from the proper base change theorem. In computations the second definition is used.[6]

6.8. Let $\mathcal{M}$ be a factorizable sheaf supported at $c \in X/Y$, $\lambda = \lambda(\mathcal{M})$. For $\nu \in Y^+$, define a vector space $\Phi(\mathcal{M})_{\lambda-\nu}$ by

$$\Phi(\mathcal{M})_{\lambda-\nu} = \Phi(\mathcal{M}^\nu).$$

If $\mu \in X,\ \mu \not\leq \lambda$, set $\Phi(\mathcal{M})_\mu = 0$. One sees easily that this way we get an exact functor Φ from $\widetilde{\mathcal{FS}}_c$ to the category of X-graded vector spaces with finite dimensional components. We extend it to the whole category $\widetilde{\mathcal{FS}}$ by additivity.

6.9. Let $\mathcal{M} \in \widetilde{\mathcal{FS}}_c$, $\lambda = \lambda(\mathcal{M})$, $\nu = \sum \nu_i i \in Y^+$. Let $i \in I$ be such that $\nu_i > 0$. Pick an unfolding of ν, $\pi : J \longrightarrow I$. For each $j \in \pi^{-1}(i)$, the restriction of π, $\pi_j : J - \{j\} \longrightarrow I$, is an unfolding of $\nu - i$.

For each $j \in \pi^{-1}(i)$, we have canonical and variation morphisms

$$u_j : \Phi_{F_j}(\pi^*\mathcal{M}^\nu) \rightleftharpoons \Phi_O(\pi^*\mathcal{M}^\nu) : v_j$$

(the facet F_j has been defined in 6.1). Taking their sum over $\pi^{-1}(i)$, we get the maps

$$\sum u_j :\ \oplus_{j\in\pi^{-1}(i)}\Phi_{F_j}(\pi^*\mathcal{M}^\nu) \rightleftharpoons \Phi_O(\pi^*\mathcal{M}) : \sum v_j$$

Note that the group Σ_π acts on both sides and the maps respect this action. After passing to Σ_π-invariants, we get the maps

(a) $\Phi_{F_j}(\pi^*\mathcal{M}^\nu)^{\Sigma_{\pi_j}} = (\oplus_{j'\in\pi^{-1}(i)}\Phi_{F_{j'}}(\pi^*\mathcal{M}^\nu))^{\Sigma_\pi} \rightleftharpoons \Phi_O(\pi^*\mathcal{M}^\nu)^{\Sigma_\pi} = \Phi(\mathcal{M})_\nu.$

Here $j \in \pi^{-1}(i)$ is an arbitrary element. Let us consider the space

$$F_j^\perp = \{(t_{j'}) \in D^J | t_j = r, t_{j'} \in D(r') \text{ for all } j' \neq j\}$$

where r, r' are some fixed real numbers such that $0 < r' < r < 1$. The space $F_j^\perp$ is transversal to F_j and may be identified with $D^{J-\{j\}}$. The factorization isomorphism induces the isomorphism

$$\pi^*\mathcal{M}^\nu|_{F_j^\perp} \cong \pi_j^*\mathcal{M}^{\nu-i} \otimes (\mathcal{I}^i_{\lambda-\nu+i})_{\{r\}} = \pi^*\mathcal{M}^{\nu-i}$$

which in turn induces the isomorphism

$$\Phi_{F_j}(\pi^*\mathcal{M}^\nu) \cong \Phi_O(\pi_j^*\mathcal{M}^{\nu-i}).$$

which is Σ_{π_j}-equivariant. Taking Σ_{π_j}-invariants and composing with the maps (a), we get the maps

(b) $\epsilon_i : \Phi(\mathcal{M})_{\nu-i} = \Phi_O(\pi_j^*\mathcal{M}^{\nu-i})^{\Sigma_j} \rightleftharpoons \Phi(\mathcal{M})_\nu : \theta_i$

[6]its independence of the choice of an unfolding follows from its equivalence to the first definition.

which do not depend on the choice of $j \in \pi^{-1}(i)$.

6.10. For an arbitrary $\mathcal{M} \in \widetilde{\mathcal{FS}}$, let us define the X-graded vector space $\Phi(\mathcal{M})$ as $\Phi(\mathcal{M}) = \oplus_{\lambda \in X} \Phi(\mathcal{M})_\lambda$.

6.11. **Theorem.** *The operators ϵ_i, θ_i $(i \in I)$ acting on the X-graded vector space $\Phi(\mathcal{M})$ satisfy the relations 2.11* (a), (b). $\square$

6.12. A factorizable sheaf $\mathcal{M}$ is finite iff the space $\Phi(\mathcal{M})$ is finite dimensional. The previous theorem says that Φ defines an exact functor

$$\Phi : \mathcal{FS} \longrightarrow \mathcal{C}.$$

One proves that Φ is a tensor functor.

6.13. **Example.** For every $\lambda \in X$, the factorizable sheaf $\mathcal{L}(\lambda)$ (cf. 4.8) is finite. It is an irreducible object of $\mathcal{FS}$, and every irreducible object in $\mathcal{FS}$ is isomorphic to some $\mathcal{L}(\lambda)$. We have $\Phi(\mathcal{L}(\lambda)) = L(\lambda)$.

The next theorem is the main result of the present work.

6.14. **Theorem.** *The functor Φ is an equivalence of braided tensor categories.* $\square$

6.15. **Remark.** As a consequence, the category $\mathcal{FS}$ is rigid. We do not know a geometric construction of the rigidity; it would be very interesting to find one.

Part II. Global (genus 0).

7. Cohesive local systems

7.1. From now on until the end of the paper we make the assumptions of 2.19. In the operadic notations below we partially follow [BD].

7.2. **Operad $\mathcal{D}$.** For a nonempty finite set J, let $D(J)$ denote the space whose points are J-tuples $\{x_j, \tau_j\}$ $(j \in J)$ where $x_j \in D$ and τ_j is a non-zero tangent vector at x_j, such that all points x_j are distinct.

Let $\tilde{D}(J)$ be the space whose points are J-tuples $\{\phi_j\}$ of holomorphic maps $\phi_j : D \longrightarrow D$ $(j \in J)$, each ϕ_j having the form $\phi_j(z) = x_j + \tau_j z$ $(x_j \in D, \tau_j \in \mathbb{C}^*)$, such that $\phi_j(D) \cap \phi_{j'}(D) = \emptyset$ for $j \neq j'$. We shall identify the j-tuple $\{\phi_j\}$ with the J-tuple $\{x_j, \tau_j\}$, and consider τ_j as a non-zero tangent vector from $T_{x_j}D$, thus identifying $T_{x_j}D$ with $\mathbb{C}$ using the local coordinate $z - x_j$. So, τ_j is the image under ϕ_j of the unit tangent vector at 0. We have an obvious map $p(J) : \tilde{D}(J) \longrightarrow D(J)$ which is a homotopy equivalence.

If $\rho : K \longrightarrow J$ is an epimorphic map of finite sets, the composition defines a holomorphic map

$$m(\rho) : \prod_J \tilde{D}(K_j) \times \tilde{D}(J) \longrightarrow \tilde{D}(K)$$

where $K_j := \rho^{-1}(j)$. If $L \xrightarrow{\sigma} K \xrightarrow{\rho} J$ are two epimorphisms of finite sets, the square

$$\begin{array}{ccc}
\prod_K \tilde{D}(L_k) \times \prod_J \tilde{D}(K_j) \times \tilde{D}(J) & \xrightarrow{m(\sigma)} & \prod_K \tilde{D}(L_k) \times \tilde{D}(K) \\
\prod m(\sigma_j) \downarrow & & \downarrow m(\sigma) \\
\prod_J \tilde{D}(L_j) \times \tilde{D}(J) & \xrightarrow{m(\rho\sigma)} & \tilde{D}(L)
\end{array}$$

commutes. Here $\sigma_j : L_j \longrightarrow K_j$ are induced by σ.

Let $*$ denote the one element set. The space $\tilde{D}(*)$ has a marked point, also to be denoted by $*$, corresponding to the identity map $\phi : D \longrightarrow D$.

If $\rho : J' \xrightarrow{\sim} J$ is an isomorphism, it induces in the obvious way an isomorphism $\rho^* : \tilde{D}(J) \xrightarrow{\sim} \tilde{D}(J')$ (resp. $D(J) \xrightarrow{\sim} D(J')$). The first map coincides with $m(\rho)$ restricted to $(\prod_J *) \times \tilde{D}(J)$. In particular, for each J, the group Σ_J of automorphisms of the set J, acts on the spaces $\tilde{D}(J), D(J)$.

The map $m(J \longrightarrow *)$ restricted to $* \times \tilde{D}(J)$ is the identity of $\tilde{D}(J)$.

We will denote the collection of the spaces and maps $\{\tilde{D}(J), m(\rho)\}$ by $\mathcal{D}$, and call it the *operad of disks with tangent vectors*.

7.3. **Coloured local systems over $\mathcal{D}$.** If $\rho : K \longrightarrow J$ is an epimorphism of finite sets and $\pi : K \longrightarrow X$ is a map of sets, we define the map $\rho_*\pi : J \longrightarrow X$ by $\rho_*\pi(j) = \sum_{K_j} \pi(k)$. For $j \in J$, we denote $K_j := \rho^{-1}(j)$ as above, and $\pi_j : K_j \longrightarrow X$ will denote the restriction of π.

Let us call an X-*coloured local system* $\mathcal{J}$ over $\mathcal{D}$ a collection of local systems $\mathcal{J}(\pi)$ over the spaces $\tilde{D}(J)$ given for every map $\pi : J \longrightarrow X$, J being a non-empty finite set, together with *factorization isomorphisms*

$$\phi(\rho) : \ m(\rho)^*\mathcal{J}(\pi) \xrightarrow{\sim} \boxed{\times}_J \mathcal{J}(\pi_j) \boxtimes \mathcal{J}(\rho_*\pi)$$

given for every epimorphism $\rho : K \longrightarrow J$ and $\pi : K \longrightarrow X$, which satisfy the properties (a), (b) below.

(a) *Associativity.* Given a map $\pi : L \longrightarrow X$ and a pair of epimorphisms $L \xrightarrow{\sigma} K \xrightarrow{\rho} J$, the square below commutes.

$$\begin{array}{ccc} m(\rho)^*m(\sigma)^*\mathcal{J}(\pi) & \stackrel{\phi(\sigma)}{\longrightarrow} & \boxed{\times}_K \mathcal{J}(\pi_k) \boxtimes m(\rho)^*\mathcal{J}(\sigma_*\pi) \\ \phi(\rho\sigma) \downarrow & & \downarrow \phi(\rho) \\ \boxed{\times}_J m(\sigma_j)^*\mathcal{J}(\pi_j) \boxtimes \mathcal{J}(\rho_*\sigma_*\pi) & \stackrel{\boxtimes\phi(\sigma_j)}{\longrightarrow} & \boxed{\times}_K \mathcal{J}(\pi_k) \boxtimes \boxed{\times}_J \mathcal{J}((\sigma_*\pi)_j) \boxtimes \mathcal{J}(\rho_*\sigma_*\pi) \end{array}$$

Note that $\pi_{j*}\sigma_j = (\sigma_*\pi)_j$.

For $\mu \in X$, let $\pi_\mu : * \longrightarrow X$ be defined by $\pi_\mu(*) = \mu$. The isomorphisms $\phi(\mathrm{id}_*)$ restricted to the marked points in $D(*)$, give the isomorphisms $\mathcal{J}(\pi_\mu)_* \xrightarrow{\sim} k$ (and imply that the local systems $\mathcal{J}(\pi_\mu)$ are one-dimensional).

(b) For any $\pi : J \longrightarrow X$, the map $\phi(\mathrm{id}_J)$ restricted to $(\prod_J *) \times \tilde{D}(J)$, equals $\mathrm{id}_{\mathcal{J}(\pi)}$.

The map $\phi(J \longrightarrow *)$ restricted to $* \times \tilde{D}(J)$, equals $\mathrm{id}_{\mathcal{J}(\pi)}$.

7.4. The definition above implies that the local systems $\mathcal{J}(\pi)$ are functorial with respect to isomorphisms. In particular, the action of the group Σ_π on $\tilde{D}(J)$ lifts to $\mathcal{J}(\pi)$.

7.5. **Standard local system over $\mathcal{D}$.** Let us define the "standard" local systems $\mathcal{J}(\pi)$ by a version of the construction 3.10.

We will use the notations of *loc. cit.* for the marked points and paths. We can identify $D(J) = D^{J\circ} \times (\mathbb{C}^*)^J$ where we have identified all tangent spaces T_xD ($x \in D$) with $\mathbb{C}$ using the local coordinate $z - x$.

We set $\mathcal{J}(\pi)_{x(\sigma)} = k$, the monodromies being $T_{\gamma(j',j'')^\pm} = \zeta^{\mp\pi(j')\cdot\pi(j'')}$, and the monodromy T_j corresponding to the counterclockwise circle of a tangent vector τ_j is $\zeta^{-2n\pi(j)}$. The factorization isomorphisms are defined by the same condition as in *loc. cit.*

This defines the *standard local system* $\mathcal{J}$ over $\mathcal{D}$. Below, the notation $\mathcal{J}$ will be reserved for this local system.

7.6. Set $P = \mathbb{P}^1(\mathbb{C})$. We pick a point $\infty \in P$ and choose a global coordinate $z : \mathbb{A}^1(\mathbb{C}) = P - \{\infty\} \xrightarrow{\sim} \mathbb{C}$. This gives local coordinates: $z - x$ at $x \in \mathbb{A}^1(\mathbb{C})$ and $1/z$ at ∞.

For a non-empty finite set J, let $P(J)$ denote the space of J-tuples $\{x_j, \tau_j\}$ where x_j are distinct points on P, and τ_j is a non-zero tangent vector at x_j. Let $\tilde{P}(J)$ denote the space whose points are J-tuples of holomorphic embeddings $\phi_j : D \longrightarrow P$ with non-intersecting images such that each ϕ_j is a restriction of an algebraic morphism $P \longrightarrow P$. We have the 1-jet projections $\tilde{P}(J) \longrightarrow P(J)$. We will use the notation $P(n)$ for $P(\{1, \dots, n\})$, etc.

An epimorphism $\rho : K \longrightarrow J$ induces the maps

$$m_P(\rho) : \prod_J \tilde{D}(K_j) \times \tilde{P}(J) \longrightarrow \tilde{P}(K)$$

and

$$\bar{m}_P(\rho) : \prod_J D(K_j) \times \tilde{P}(J) \longrightarrow P(K).$$

For a pair of epimorphisms $L \xrightarrow{\sigma} K \xrightarrow{\rho} J$, the square

$$\begin{array}{ccc} \prod_K \tilde{D}(L_k) \times \prod_J \tilde{D}(K_j) \times \tilde{P}(J) & \xrightarrow{m_P(\sigma)} & \prod_K \tilde{D}(L_k) \times \tilde{P}(K) \\ \prod m(\sigma_j) \downarrow & & \downarrow m_P(\sigma) \\ \prod_J \tilde{D}(L_j) \times \tilde{P}(J) & \xrightarrow{m_P(\rho\sigma)} & \tilde{P}(L) \end{array}$$

commutes.

If $\rho : J' \xrightarrow{\sim} J$ is an isomorphism, it induces in the obvious way an isomorphism $\rho^* : \tilde{P}(J) \xrightarrow{\sim} \tilde{P}(J')$ (resp. $P(J) \xrightarrow{\sim} P(J')$). This last map coincides with $m_P(\rho)$ (resp. $\bar{m}_P(\rho)$) restricted to $(\prod_J *) \times \tilde{P}(J)$ (resp. $(\prod_J *) \times P(J)$).

The collection of the spaces and maps $\{\tilde{P}(J), m_P(\rho)\}$ form an object $\tilde{P}$ called a *right module* over the operad $\mathcal{D}$.

7.7. Fix an element $\mu \in X$. Let us say that a map $\pi : J \longrightarrow X$ has *level* μ if $\sum_J \pi(j) = \mu$.

A *cohesive local system* $\mathcal{H}$ *of level* μ over P is a collection of local systems $\mathcal{H}(\pi)$ over $\tilde{P}(J)$ given for every $\pi : J \longrightarrow X$ of level μ, together with *factorization isomorphisms*

$$\phi_P(\rho) : m_P(\rho)^* \mathcal{H}(\pi) \xrightarrow{\sim} \boxed{\times}_J \mathcal{J}(\pi_j) \boxtimes \mathcal{H}(\rho_* \pi)$$

given for every epimorphism $\rho : K \longrightarrow J$ and $\pi : K \longrightarrow X$ of level μ, which satisfy the properties (a), (b) below.

(a) *Associativity.* Given a map $\pi : L \longrightarrow X$ and a pair of epimorphisms $L \xrightarrow{\sigma} K \xrightarrow{\rho} J$, the square below commutes.

$$\begin{array}{ccc} m_P(\rho)^* m_P(\sigma)^* \mathcal{H}(\pi) & \xrightarrow{\phi_P(\sigma)} & \boxed{\times}_K \mathcal{J}(\pi_k) \boxtimes m_P(\rho)^* \mathcal{H}(\sigma_* \pi) \\ \phi_P(\rho\sigma) \downarrow & & \downarrow \phi_P(\rho) \\ \boxed{\times}_J m(\sigma_j)^* \mathcal{J}(\pi_j) \boxtimes \mathcal{H}(\rho_* \sigma_* \pi) & \xrightarrow{\boxtimes \phi(\sigma_j)} & \boxed{\times}_K \mathcal{J}(\pi_k) \boxtimes \boxed{\times}_J \mathcal{J}((\sigma_* \pi)_j) \boxtimes \mathcal{H}(\rho_* \sigma_* \pi) \end{array}$$

(b) For any $\pi : J \longrightarrow X$, the map $\phi_P(\mathrm{id}_J)$ restricted to $(\prod_J *) \times \tilde{P}(J)$, equals $\mathrm{id}_{\mathcal{H}(\pi)}$.

7.8. The definition above implies that the local systems $\mathcal{H}(\pi)$ are functorial with respect to isomorphisms. In particular, the action of the group Σ_π on $\tilde{P}(J)$, lifts to $\mathcal{H}(\pi)$.

7.9. **Theorem.** *For each $\mu \in X$ such that $\mu \equiv 2\rho_\ell \ \mathrm{mod} Y_\ell$, there exists a unique up to an isomorphism one-dimensional cohesive local system $\mathcal{H} = \mathcal{H}^{(\mu)}$ of level μ over P.* $\square$

The element $\rho_\ell \in X$ is defined in 2.2 and the lattice Y_ℓ in 2.3.

¿From now on, let us fix such a local system $\mathcal{H}^{(\mu)}$ for each μ as in the theorem.

7.10. Note that the obvious maps $p(J) : \tilde{P}(J) \longrightarrow P(J)$ are homotopy equivalences. Therefore the local systems $\mathcal{H}(\pi)$ $(\pi : J \longrightarrow X)$ descend to the unique local systems over $P(J)$, to be denoted by the same letter.

8. Gluing

8.1. Let us fix a finite set K. For $\nu \in Y^+$, pick an unfolding of ν, $\pi : J \longrightarrow I$. Consider the space P^ν; its points are formal linear combinations $\sum_J \pi(j)x_j$ $(x_j \in P)$. We define the space $P^\nu(K) = P(K) \times P^\nu$; its points are tuples $\{y_k, \tau_k, \sum \pi(j)x_j\}$ $(k \in K, y_k \in P, \tau_k \neq 0$ in $T_{y_k}P)$, all y_k being distinct. Let $P^\nu(K)^\circ$ (resp. $P^\nu(K)^\bullet$) be the subspace whose points are tuples as above with all x_j distinct from y_k and pairwise distinct (resp. all x_j distinct from y_k). We will use the notation $P^\nu(n)$ for $P^\nu(\{1, \dots, n\})$, etc.

Let $\nu' \in Y^+$ and $\vec{\nu} = \{\nu_k\} \in (Y^+)^K$ be such that $\sum_K \nu_k + \nu' = \nu$. Define the space $P^{\vec{\nu};\nu'} \subset \tilde{P}(K) \times P^\nu$ consisting of tuples $\{\phi_k, \sum \pi(j)x_j\}$ such that for each k, ν_k of the points x_j lie inside $\phi_k(D)$ and ν' of them lie outside all closures of these disks.

Let $\vec{0} \in (Y^+)^K$ be the zero K-tuple. Define the space $\tilde{P}^\nu(K) := P^{\vec{0};\nu}$. We have obvious maps

(a) $\prod_K D^{\nu_k} \times \tilde{P}^{\nu'}(K) \overset{p(\vec{\nu};\nu')}{\longleftarrow} P^{\vec{\nu};\nu'} \overset{m(\vec{\nu};\nu')}{\longrightarrow} P^\nu(K)$.

Let $\vec{\nu}^1, \vec{\nu}^2 \in (Y^+)^K$ be such that $\vec{\nu}^1 + \vec{\nu}^2 = \vec{\nu}$. Define the space

$P^{\vec{\nu}^1;\vec{\nu}^2;\nu} \subset \tilde{P}(K) \times \mathbb{R}^K_{>0} \times \tilde{P}(K) \times P^\nu$ consisting of all tuples $\{\phi_k; r_k; \phi'_k; \sum \pi(j)x_j\}$ such that $r_k < 1; \phi_k(z) = \phi'_k(r_k z)$, and ν^1_k (resp. ν^2_k, ν') from the points x_j lie inside $\phi_k(D)$ (resp. inside the annulus $\phi'_k(D) - \overline{\phi_k(D)}$, inside $P - \bigcup \overline{\phi'_k(D)}$). We set $\tilde{P}^{\vec{\nu};\nu'} := P^{\vec{0};\vec{\nu};\nu'}$, cf. 5.2.

We have a commutative romb (cf. 3.5):

(b)

$$\begin{tikzcd}
 & & P^{\nu}(K) & & \\
 & P^{\vec{\nu}^1;\nu^2+\nu'} \arrow[ur, "m"] \arrow[dl, "p"'] & & P^{\vec{\nu}^1+\vec{\nu}^2;\nu'} \arrow[ul, "m"'] \arrow[dr, "p"] & \\
\prod D^{\nu^1_k}\times \tilde{P}^{\nu^2+\nu'}(K) & & P^{\vec{\nu}^1;\vec{\nu}^2;\nu'} \arrow[ul, "m"'] \arrow[ur, "m"] \arrow[dl, "p"'] \arrow[dr, "p"] & & \prod D^{\nu^1_k+\nu^2_k}\times \tilde{P}^{\nu'}(K) \\
 & \prod D^{\nu^1_k}\times \tilde{P}^{\vec{\nu}^2;\nu'} \arrow[ul, "m"] \arrow[dr, "p"] & & \prod D^{\nu^1_k,\nu^2_k}\times \tilde{P}^{\nu'}(K) \arrow[ur, "m"'] \arrow[dl, "p"'] & \\
 & & \prod D^{\nu^1_k}\times \prod A^{\nu^2_k}\times \tilde{P}^{\nu'}(K) & &
\end{tikzcd}$$

Here $\nu^2 := \sum \nu^2_k$.

8.2. Let $P(K;J)$ be the space consisting of tuples $\{y_k;\tau_k;x_j\}$ $(k\in K, j\in J, y_k, x_j \in P, \tau_k\neq 0$ in $T_{y_k}P)$, where all points x_k, y_j are distinct. We have $P^{\nu}(K)^{\circ} = P(K;J)/\Sigma_{\pi}$. We have an obvious projection $P(K\coprod J)\longrightarrow P(K;J)$.

8.3. Let $\{\mathcal{M}_k\}$ be a K-tuple of factorizable sheaves supported at some cosets in X/Y; let $\mu_k = \lambda(\mathcal{M}_k)$.

Let $\tilde{\pi} : K\coprod J \longrightarrow X$ be a map defined by $\tilde{\pi}(k) = \mu_k$, $\tilde{\pi}(j) = -\pi(j)\in I\hookrightarrow X$. The local system $\mathcal{H}(\tilde{\pi})$ over $P(K\coprod J)$ descends to $P(K;J)$ since $\zeta^{2n(-i)} = 1$ for all $i\in I$; this one in turn descends to the unique local system $\tilde{\mathcal{H}}^{\nu}_{\vec{\mu}}$ over $P^{\nu}(K)^{\circ}$, due to Σ_{π}-equivariance. Let us define the local system $\mathcal{H}^{\nu}_{\vec{\mu}} := \tilde{\mathcal{H}}^{\nu}_{\vec{\mu}}\otimes \mathcal{S}ign^{\nu}$. Here $\mathcal{S}ign^{\nu}$ denotes the inverse image of the sign local system on $P^{\nu\circ}$ (defined in the same manner as for the disk, cf. 3.9) under the forgetful map $P^{\nu}(K)^{\circ}\longrightarrow P^{\nu\circ}$.

Let $\mathcal{H}^{\nu\bullet}_{\vec{\mu}}$ be the perverse sheaf over $P^{\nu}(K)^{\bullet}$ which is the middle extension of $\mathcal{H}^{\nu}_{\vec{\mu}}[\dim P^{\nu}(K)]$. Let us denote by the same letter the inverse image of this perverse sheaf on the space $\tilde{P}^{\nu}(K)$ with respect to the evident projection $\tilde{P}^{\nu}(K)\longrightarrow P^{\nu}(K)^{\bullet}$.

8.4. Let us call an element $\nu\in Y^+$ *admissible* (for a K-tuple $\{\mu_k\}$) if $\sum\mu_k - \nu \equiv 2\rho_{\ell} \operatorname{mod} Y_{\ell}$, see 2.3.

8.5. **Theorem - definition.** *For each admissible ν, there exists a unique, up to a unique isomorphism, perverse sheaf, denoted by $\boxed{\times}_K^{(\nu)}\ \mathcal{M}_k$, over $P^{\nu}(K)$, equipped with isomorphisms*

$$\psi(\vec{\nu};\nu') : m^*\boxed{\times}_K^{(\nu)}\ \mathcal{M}_k \xrightarrow{\sim} p^*(\boxed{\times}_K\ \mathcal{M}_k^{\nu_k}\boxtimes \mathcal{H}^{\nu'\bullet}_{\vec{\mu}-\vec{\nu}})$$

given for every diagram 8.1 (a) such that for each rhomb 8.1 (b) the cocycle condition

$$\phi(\vec{\nu}^2;\nu')\circ\psi(\vec{\nu}^1;\nu^2+\nu') = (\boxed{\times}_K\ \psi^{\mathcal{M}_k}(\nu^1_k,\nu^2_k))\circ\psi(\vec{\nu}^1+\vec{\nu}^2;\nu')$$

holds. $\square$

8.6. The sheaf $\boxed{\times}_K^{(\nu)} \mathcal{M}_k$ defines for each K-tuple of $\vec{y} = \{y_k, \tau_k\}$ of points of P with non-zero tangent vectors, the sheaf $\boxed{\times}_{\vec{y}}^{(\nu)} \mathcal{M}_k$ over P^ν, to be called the *gluing of the factorizable sheaves* $\mathcal{M}_k$ *into the points* (y_k, τ_k).

8.7. **Example.** The sheaf $\boxed{\times}_K^{(\nu)} \mathcal{L}(\mu_k)$ is equal to the middle extension of the sheaf $\mathcal{H}_{\vec{\mu}}^{\nu\bullet}$.

9. Semiinfinite cohomology

In this section we review the theory of semiinfinite cohomology in the category $\mathcal{C}$, due to S. M. Arkhipov, cf. [Ark].

9.1. Let $\mathcal{C}_r$ be a category whose objects are *right* $\mathfrak{u}$-modules N, finite dimensional over k, with a given X-grading $N = \oplus_{\lambda \in X} N_\lambda$ such that $xK_\nu = \zeta^{-\langle \nu, \lambda \rangle} x$ for any $\nu \in Y, \lambda \in X, x \in N_\lambda$. All definitions and results concerning the category $\mathcal{C}$ given above and below, have the obvious versions for the category $\mathcal{C}_r$.

For $M \in \mathcal{C}$, define $M^\vee \in \mathcal{C}_r$ as follows: $(M^\vee)_\lambda = (M_{-\lambda})^*$ (the dual vector space); the action of the operators θ_i, ϵ_i being the transpose of their action on M. This way we get an equivalence ${}^\vee : \mathcal{C}^{\mathrm{opp}} \xrightarrow{\sim} \mathcal{C}_r$.

9.2. Let us call an object $M \in \mathcal{C}$ $\mathfrak{u}^-$- (resp. $\mathfrak{u}^+$-) *good* if it admits a filtration whose successive quotients have the form $\mathrm{ind}_{\mathfrak{u}_{\geq 0}}^{\mathfrak{u}}(M')$ (resp. $\mathrm{ind}_{\mathfrak{u}_{\leq 0}}^{\mathfrak{u}}(M''))$ for some $M' \in \mathcal{C}^{\geq 0}$ (resp. $M'' \in \mathcal{C}^{\leq 0}$) (cf. 2.16). These classes of objects are stable with respect to the tensor multiplication by an arbitrary object of $\mathcal{C}$.

If M is $\mathfrak{u}^-$- (resp. $\mathfrak{u}^+$-) good then M^* is $\mathfrak{u}^-$- (resp. $\mathfrak{u}^+$-) good.

If M is $\mathfrak{u}^-$-good and M' is $\mathfrak{u}^+$-good then $M \otimes M'$ is a projective object in $\mathcal{C}$.

9.3. Let us say that a complex $M^\bullet$ in $\mathcal{C}$ is *concave* (resp. *convex*) if

(a) there exists $\mu \in X$ such that all nonzero components $M^\bullet_\lambda$ have the weight $\lambda \geq \mu$ (resp. $\lambda \leq \mu$);

(b) for any $\lambda \in X$, the complex $M^\bullet_\lambda$ is finite.

9.4. For an object $M \in \mathcal{C}$, we will call a *left* (resp. *right*) $\mathfrak{u}^\pm$-*good resolution* of M an exact complex

$$\ldots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

(resp.

$$0 \longrightarrow M \longrightarrow R^0 \longrightarrow R^1 \longrightarrow \ldots)$$

such that all P^i (resp. R^i) are $\mathfrak{u}^\pm$-good.

9.5. **Lemma.** *Each object $M \in \mathcal{C}$ admits a convex $\mathfrak{u}^-$-good left resolution, a concave $\mathfrak{u}^+$-good left resolution, a concave $\mathfrak{u}^-$-good right resolution and a convex $\mathfrak{u}^+$-good right resolution.* $\square$

9.6. For $N \in \mathcal{C}_r, M \in \mathcal{C}$, define a vector space $N \otimes_{\mathcal{C}} M$ as the zero weight component of the tensor product $N \otimes_{\mathfrak{u}} M$ (which has an obvious X-grading).

For $M, M' \in \mathcal{C}$, we have an obvious perfect pairing

(a) $\mathrm{Hom}_{\mathcal{C}}(M, M') \otimes (M'^{\vee} \otimes_{\mathcal{C}} M) \longrightarrow k$.

9.7. $M, M' \in \mathcal{C}, N \in \mathcal{C}_r$ and $n \in \mathbb{Z}$, define the *semiinifinite* Ext *and* Tor spaces

$$\mathrm{Ext}_{\mathcal{C}}^{\frac{\infty}{2}+n}(M, M') = H^n(\mathrm{Hom}_{\mathcal{C}}(R^{\bullet}_{\searrow}(M), R^{\bullet}_{\nearrow}(M')))$$

where $R^{\bullet}_{\searrow}(M)$ (resp. $R^{\bullet}_{\nearrow}(M')$) is an arbitrary $\mathfrak{u}^+$-good convex right resolution of M (resp. $\mathfrak{u}^-$-good concave right resolution of M'),

$$\mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}+n}(N, M) = H^{-n}(P^{\bullet}_{\swarrow}(N) \otimes_{\mathcal{C}} R^{\bullet}_{\searrow}(M))$$

where $P^{\bullet}_{\swarrow}(N)$ is an arbitrary $\mathfrak{u}^-$-good convex left resolution of N.

This definition does not depend, up to a unique isomorphism, upon the choice of resolutions, and is functorial.

These spaces are finite dimensional and are non-zero only for finite number of degrees n.

The pairing 9.6 (a) induces perfect pairings

$$\mathrm{Ext}_{\mathcal{C}}^{\frac{\infty}{2}+n}(M, M') \otimes \mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}+n}(M'^{\vee}, M) \longrightarrow k \ (n \in \mathbb{Z}).$$

10. Conformal blocks (genus 0)

In this section we suppose that $k = \mathbb{C}$.

10.1. Let $M \in \mathcal{C}$. We have a canonical embedding of vector spaces

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, M) \hookrightarrow M$$

which identifies $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, M)$ with the maximal trivial subobject of M. Here "trivial" means "isomorphic to a sum of a few copies of the object $\mathbf{1}$". Dually, we have a canonical epimorhism

$$M \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, \mathbf{1})^*$$

which identifies $\mathrm{Hom}_{\mathcal{C}}(M, \mathbf{1})^*$ with the maximal trivial quotient of M. Let us denote by $\langle M \rangle$ the image of the composition

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, M) \longrightarrow M \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, \mathbf{1})^*$$

Thus, $\langle M \rangle$ is canonically a subquotient of M.

One sees easily that if $N \subset M$ is a trivial direct summand of M which is maximal, i.e. not contained in greater direct summand, then we have a canonical isomorphism $\langle M\rangle \overset{\sim}{\longrightarrow} N$. For this reason, we will call $\langle M\rangle$ *the maximal trivial direct summand* of M.

10.2. Let $\gamma_0 \in Y$ denote the highest coroot and $\beta_0 \in Y$ denote the coroot dual to the highest root ($\gamma_0 = \beta_0$ for a simply laced root datum).

Let us define the *first alcove* $\Delta_\ell \subset X$ by

$$\Delta_\ell = \{\lambda \in X |\ \langle i, \lambda+\rho\rangle > 0 \text{ for all } i \in I;\ \langle \gamma_0, \lambda+\rho\rangle < \ell\}$$

if d does not divide ℓ, i.e. if $\ell_i = \ell$ for all $i \in I$, and by

$$\Delta_\ell = \{\lambda \in X |\ \langle i, \lambda+\rho\rangle > 0 \text{ for all } i \in I;\ \langle \beta_0, \lambda+\rho\rangle < \ell_{\beta_0}\}$$

otherwise, cf. [AP] 3.19 (d is defined in 2.1, and ℓ_{β_0} in 2.2). Note that $\ell_{\beta_0} = \ell/d$.

10.3. For $\lambda_1, \ldots, \lambda_n \in \Delta_\ell$, define the *space of conformal blocks* by

$$\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle := \langle L(\lambda_1) \otimes \ldots \otimes L(\lambda_n)\rangle.$$

In fact, due to the ribbon structure on $\mathcal{C}$, the right hand side is a *local system* over the space $P(n) := P(\{1, \ldots, n\})$ (cf. [D1]). It is more appropriate to consider the previous equality as the definition of the *local system of conformal blocks* over $P(n)$.

10.4. **Theorem.** (Arkhipov) *For each $\lambda_1, \ldots, \lambda_n \in \Delta_\ell$, the space of conformal blocks $\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle$ is naturally a subquotient of the space*

$\mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}+0}(\mathbf{1}_r, L(\lambda_1) \otimes \ldots \otimes L(\lambda_n) \otimes L(2\rho_\ell))$.

More precisely, due to the ribbon structure on $\mathcal{C}$, the latter space is a stalk of a local system over $P(n+1)$, and inverse image of the local system $\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle$ under the projection onto the first coordinates $P(n+1) \longrightarrow P(n)$, is a natural subquotient of this local system.

Here $\mathbf{1}_r$ is the unit object in $\mathcal{C}_r$.

Examples, also due to Arkhipov, show that the local systems of conformal blocks are in general *proper* subquotients of the corresponding Tor local systems.

This theorem is an immediate consequence of the next lemma, which in turn follows from the geometric theorem 11.2 below, cf. 11.3.

10.5. **Lemma.** *We have* $\mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}+n}(\mathbf{1}_r, L(2\rho_\ell)) = k$ *if* $n = 0$, *and* 0 *otherwise.* $\Box$

10.6. Let $\mathfrak{g}$ be the simple Lie algebra (over k) associated with our Cartan datum; let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra.

Let $\mathcal{MS}$ denote the category of integrable $\hat{\mathfrak{g}}$-modules of central charge $\kappa - h$. Here κ is a fixed positive integer, h is the dual Coxeter number of our Cartan datum. $\mathcal{MS}$ is a semisimple abelian category whose irreducible objects are $\mathfrak{L}(\lambda),\ \lambda \in \Delta_\ell$ where $l = 2d\kappa$, i.e. $\ell = d\kappa$ (we are grateful to Shurik Kirillov who pointed out the necessity of the factor d here) and $\mathfrak{L}(\lambda)$ is the highest weight module with a highest vector v whose "top" part $\mathfrak{g} \cdot v$ is the irreducible $\mathfrak{g}$-module of the highest weight λ.

According to Conformal field theory, $\mathcal{MS}$ has a natural structure of a ribbon category, cf. [MS], [K].

The usual local systems of conformal blocks in the WZW model may be defined as

$$\langle \mathfrak{L}(\lambda_{\mathfrak{1}}), \ldots, \mathfrak{L}(\lambda_{\mathfrak{n}}) \rangle = \mathrm{Hom}_{\mathcal{MS}}(\mathbf{1}, \mathfrak{L}(\lambda_{\mathfrak{1}}) \otimes \ldots \otimes \mathfrak{L}(\lambda_{\mathfrak{n}}))$$

the structure of a local system on the right hand side is due to the ribbon structure on $\mathcal{MS}$.

10.7. Let $\zeta = \exp(\pi\sqrt{-1}/d\kappa)$. We have an exact functor

$$\phi :\ \mathcal{MS} \longrightarrow \mathcal{C}$$

sending $\mathfrak{L}(\lambda)$ to $L(\lambda)$. This functor identifies $\mathcal{MS}$ with a full subcategory of $\mathcal{C}$.

The functor ϕ does not respect the tensor structures. It admits the left and right adjoints, $\phi^\flat, \phi^\sharp$. For $M \in \mathcal{C}$, let $\langle M \rangle_{\mathcal{MS}}$ denotes the image of the composition

$$\phi \circ \phi^\flat(M) \longrightarrow M \longrightarrow \phi \circ \phi^\sharp(M).$$

We have the following comparison theorem.

10.8. **Theorem.** *We have naturally*

$$\phi(\mathfrak{M} \otimes \mathfrak{M}') = \langle \phi(\mathfrak{M}) \otimes \phi(\mathfrak{M}') \rangle_{\mathcal{MS}}.$$

This follows from the combination of the results of [AP], [KL], [L3] and [F].

10.9. **Corollary.** *For any $\lambda_1, \ldots, \lambda_n \in \Delta_\ell$, the functor ϕ induces an isomorphism of local systems*

$$\langle \mathfrak{L}(\lambda_{\mathfrak{1}}), \ldots, \mathfrak{L}(\lambda_{\mathfrak{n}}) \rangle = \langle \mathfrak{L}(\lambda_{\mathfrak{1}}), \ldots, \mathfrak{L}(\lambda_{\mathfrak{n}}) \rangle. \ \Box$$

11. Integration

We keep the notations of the previous section.

11.1. Let K be a finite set, $m = \mathrm{card}(K)$, $\{\mathcal{M}_k\}$ a K-tuple of finite factorizable sheaves, $\mathcal{M}_k \in \mathcal{FS}_{c_k}$, $\mu_k := \lambda(\mathcal{M}_k)$. Assume that $\nu := \sum_K \ \mu_k - 2\rho_\ell$ belongs to Y^+.

Let $\eta : P^\nu(K) \longrightarrow P(K)$ be the projection.

11.2. **Theorem.** *We have canonical isomorphisms of local systems over $P(K)$*

$$R^{a-2m}\eta_*(\boxed{\times}_K^{(\nu)} \ \mathcal{M}_k) = \mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}-a}(\mathbf{1}_{\mathrm{r}}, \otimes_{\mathrm{K}} \ \Phi(\mathcal{M}_{\mathrm{k}})) \ (\mathrm{a} \in \mathbb{Z}),$$

the structure of a local system on the right hand side being induced by the ribbon structure on $\mathcal{C}$. $\square$

11.3. **Proof of Lemma 10.5.** Apply the previous theorem to the case when the K-tuple consists of one sheaf $\mathcal{L}(2\rho_\ell)$ and $\nu = 0$. $\square$

11.4. From now until the end of the section, $k = \mathbb{C}$ and $\zeta = \exp(\pi\sqrt{-1}/d\kappa)$. Let $\lambda_1, \ldots, \lambda_n \in \Delta_\ell$. Let $\nu = \sum_{m=1}^n \ \lambda_m$; assume that $\nu \in Y^+$. Set $\vec{\mu} = \{\lambda_1, \ldots, \lambda_n, 2\rho_\ell\}$.

Let η be the projection $P^\nu(n+1) := P^\nu(\{1, \ldots, n+1\}) \longrightarrow P(n+1)$ and $p : P(n+1) \longrightarrow P(n)$ be the projection on the first coordinates.

Let $\mathcal{H}^{\nu\sharp}_{\vec{\mu}}$ denote the middle extension of the sheaf $\mathcal{H}^{\nu\bullet}_{\vec{\mu}}$. By the Example 8.7,

$$\mathcal{H}^{\nu\sharp}_{\vec{\mu}} = \boxed{\times}^{(\nu)}_{1\le a\le n}\mathcal{L}(\lambda_a) \boxtimes \mathcal{L}(2\rho_\ell).$$

11.5. **Theorem.** *The local system $p^*\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle$ is canonically a subquotient of the local system*

$$R^{-2n-2}\eta_*\mathcal{H}^{\nu\sharp}_{\vec{\mu}}.$$

This theorem is an immediate corollary of the previous one and of the Theorem 10.4.

11.6. **Corollary.** *In the notations of the previous theorem, the local system $\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle$ is semisimple.*

Proof. The local system $p^*\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle$ is a subquotient of the geometric local system $R^{-2n-2}\eta_*\mathcal{H}^{\nu\sharp}_{\vec{\mu}}$, and hence is semisimple by the Beilinson-Bernstein-Deligne-Gabber Decomposition theorem, [BBD], Théorème 6.2.5. Therefore, the local system $\langle L(\lambda_1), \ldots, L(\lambda_n)\rangle$ is also semisimple, since the map p induces the surjection on the fundamental groups. $\square$

11.7. For a sheaf $\mathcal{F}$, let $\bar{\mathcal{F}}$ denote the sheaf obtained from $\mathcal{F}$ by the complex conjugation on the coefficients.

If a perverse sheaf $\mathcal{F}$ on P^ν is obtained by gluing some irreducible factorizable sheaves into some points of P then its Verdier dual $D\mathcal{F}$ is canonically isomorphic to $\bar{\mathcal{F}}$. Therefore, the Poincaré-Verdier duality induces a perfect hermitian pairing on $R\Gamma(P^\nu;\mathcal{F})$.

Therefore, in notations of theorem 11.5, The Poincaré-Verdier duality induces a non-degenerate hermitian form on the local system $R^{-2n-2}\eta_*\mathcal{H}^{\nu\sharp}_{\vec{\mu}}$.

By a little more elaborated argument using fusion, one can introduce a canonical hermitian form on the systems of conformal blocks.

Compare [K], where a certain hermitian form on the spaces of conformal blocks (defined up to a positive constant) has been introduced.

11.8. By the similar reasons, the Verdier duality defines a hermitian form on all irreducible objects of $\mathcal{C}$ (since the Verdier duality commutes with Φ, cf. Theorem 6.3).

12. Regular representation

12.1. From now on we are going to modify slightly the definition of the categories $\mathcal{C}$ and $\mathcal{FS}$. Let X_ℓ be the lattice

$$X_\ell = \{\mu \in X\otimes\mathbb{Q}|\ \mu\cdot\mathbb{Y}_\ell \in \ell\mathbb{Z}\}$$

We have obviously $X\subset X_\ell$, and $X = X_\ell$ if $d|\ell$.

In this part we will denote by $\mathcal{C}$ a category of *X_ℓ-graded* (instead of X-graded) finite dimensional vector spaces $M=\oplus_{\lambda\in X_\ell}M_\lambda$ equipped with linear operators $\theta_i: M_\lambda \longrightarrow M_{\lambda-i'}$, $\epsilon_i: M_\lambda\longrightarrow M_{\lambda+i'}$ which satisfy the relations 2.11 (a), (b). This makes sense since $\langle d_i i,\lambda\rangle = i'\cdot\lambda\in\mathbb{Z}$ for each $i\in I, \lambda\in X_\ell$.

Also, in the definition of $\mathcal{FS}$ we replace X by X_ℓ. All the results of the previous parts hold true *verbatim* with this modification.

We set $\mathrm{d}_\ell = \mathrm{card}(X_\ell/Y_\ell)$; this number is equal to the determinant of the form $\mu_1,\mu_2\mapsto \frac{1}{\ell}\mu_1\cdot\mu_2$ on Y_ℓ.

12.2. Let $\tilde{\mathfrak{u}}\subset\mathfrak{u}$ be the k-subalgebra generated by $\tilde{K}_i,\epsilon_i,\theta_i$ $(i\in I)$. Following the method of [L1] 23.1, define a new algebra $\dot{\mathfrak{u}}$ (without unit) as follows.

If $\mu',\mu''\in X_\ell$, we set

$$_{\mu'}\tilde{\mathfrak{u}}_{\mu''} = \tilde{\mathfrak{u}}/(\sum_{i\in I}(\tilde{K}_i-\zeta^{i\cdot\mu'})\tilde{\mathfrak{u}} + \sum_{i\in I}\tilde{\mathfrak{u}}(\tilde{K}_i-\zeta^{i\cdot\mu''}));\ \dot{\mathfrak{u}} = \oplus_{\mu',\mu''\in X_\ell}(_{\mu'}\tilde{\mathfrak{u}}_{\mu''}).$$

Let $\pi_{\mu',\mu''}:\tilde{\mathfrak{u}}\longrightarrow\ {}_{\mu'}\tilde{\mathfrak{u}}_{\mu''}$ be the canonical projection. We set $1_\mu = \pi_{\mu,\mu}(1)\in\dot{\mathfrak{u}}$. The structure of an algebra on $\dot{\mathfrak{u}}$ is defined as in *loc. cit.*

As in *loc. cit.*, the category $\mathcal{C}$ may be identified with the category of finite dimensional (over k) (left) $\dot{\mathfrak{u}}$-modules M which are *unital*, i.e.

(a) for every $x \in M$, $\sum_{\mu\in X_\ell} 1_\mu x = x$.

If M is such a module, the X_ℓ-grading on M is defined by $M_\mu = 1_\mu M$.

Let $\mathfrak{u}'$ denote the quotient algebra of the algebra $\tilde{\mathfrak{u}}$ by the relations $\tilde{K}_i^{l_i} = 1$ $(i \in I)$. Here $l_i := \frac{l}{(l,d_i)}$. We have an isomorphism of vector spaces $\dot{\mathfrak{u}} = \mathfrak{u}' \otimes k[Y_\ell]$, cf. 12.5 below.

12.3. Let $a : \mathcal{C} \xrightarrow{\sim} \mathcal{C}_r$ be an equivalence defined by $aM = M$ $(M \in \mathcal{C})$ as an X_ℓ-vector space, $mx = A(x)m$ $(x \in \mathfrak{u}, m \in M)$. Here $A : \mathfrak{u} \longrightarrow \mathfrak{u}^{\mathrm{opp}}$ is the antipode. We will use the same notation a for a similar equivalence $\mathcal{C}_r \xrightarrow{\sim} \mathcal{C}$.

Let us consider the category $\mathcal{C}\otimes\mathcal{C}$ (resp. $\mathcal{C}\otimes\mathcal{C}_r$) which may be identified with the category of finite dimensional $\dot{\mathfrak{u}} \otimes \dot{\mathfrak{u}}$- (resp. $\dot{\mathfrak{u}} \otimes (\dot{\mathfrak{u}})^{\mathrm{opp}}$-) modules satisfying a "unitality" condition similar to (a) above. Let us consider the algebra $\dot{\mathfrak{u}}$ itself as a regular $\dot{\mathfrak{u}} \otimes (\dot{\mathfrak{u}})^{\mathrm{opp}}$-module. It is infinite dimensional, but is a union of finite dimensional modules, hence it may be considered as an object of the category $\mathrm{Ind}(\mathcal{C}\otimes\mathcal{C}_r) = \mathrm{Ind}\mathcal{C}\otimes\mathrm{Ind}\mathcal{C}_r$ where Ind denotes the category of Ind-objects, cf. [D4] §4. Let us denote by $\mathbf{R}$ the image of this object under the equivalence $\mathrm{Id} \otimes a : \mathrm{Ind}\mathcal{C} \otimes \mathrm{Ind}\mathcal{C}_r \xrightarrow{\sim} \mathrm{Ind}\mathcal{C} \otimes \mathrm{Ind}\mathcal{C}$.

Every object $\mathcal{O} \in \mathcal{C} \otimes \mathcal{C}$ induces a functor $F_{\mathcal{O}} : \mathcal{C} \longrightarrow \mathcal{C}$ defined by

$$F_{\mathcal{O}}(M) = a(aM \otimes_{\mathcal{C}} \mathcal{O}).$$

The same formula defines a functor $F_{\mathcal{O}} : \mathrm{Ind}\mathcal{C} \longrightarrow \mathrm{Ind}\mathcal{C}$ for $\mathcal{O} \in \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C})$.

We have $F_{\mathbf{R}} = \mathrm{Id}_{\mathrm{Ind}\mathcal{C}}$.

We can consider a version of the above formalism using semiinfinite Tor's. An object $\mathcal{O} \in \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C})$ defines functors $F_{\mathcal{O};\frac{\infty}{2}+n} : \mathrm{Ind}\mathcal{C} \longrightarrow \mathrm{Ind}\mathcal{C}$ $(n \in \mathbb{Z})$ defined by

$$F_{\mathcal{O};\frac{\infty}{2}+n}(M) = a\mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}+n}(aM, \mathcal{O}).$$

12.4. **Theorem.** (i) *We have* $F_{\mathbf{R};\frac{\infty}{2}+n} = \mathrm{Id}_{\mathrm{Ind}\mathcal{C}}$ *if* $n = 0$, *and* 0 *otherwise.*

(ii) *Conversely, suppose we have an object* $Q \in \mathrm{Ind}(\mathcal{C}\otimes\mathcal{C})$ *together with an isomorphism of functors* $\phi : F_{\mathbf{R};\frac{\infty}{2}+\bullet} \xrightarrow{\sim} F_{Q;\frac{\infty}{2}+\bullet}$. *Then* ϕ *is induced by the unique isomorphism* $\mathbf{R} \xrightarrow{\sim} Q$.
$\Box$

12.5. **Adjoint representation.** For $\mu \in Y_\ell$, let $T(\mu)$ be a one-dimensional $\dot{\mathfrak{u}} \otimes (\dot{\mathfrak{u}})^{\mathrm{opp}}$-module equal to $L(\mu)$ (resp. to $aL(-\mu)$) as a $\dot{\mathfrak{u}}$- (resp. $(\dot{\mathfrak{u}})^{\mathrm{opp}}$-) module. Let us consider the module $T_\mu\mathbf{R} = \mathbf{R}\otimes T(\mu) \in \mathrm{Ind}(\mathcal{C}\otimes\mathcal{C})$. This object represents the same functor $\mathrm{Id}_{\mathrm{Ind}\mathcal{C}}$, hence we have a canonical isomorphism $t_\mu : \mathbf{R} \xrightarrow{\sim} T_\mu\mathbf{R}$.

Let us denote by $\hat{\mathbf{ad}} \in \mathrm{Ind}\mathcal{C}$ the image of $\mathbf{R}$ under the tensor product $\otimes : \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}) \longrightarrow \mathrm{Ind}\mathcal{C}$. The isomorphisms t_μ above induce an action of the lattice Y_ℓ on $\hat{\mathbf{ad}}$. Set

$\mathbf{ad} = \hat{\mathbf{ad}}/Y_\ell$. This is an object of $\mathcal{C} \subset \mathrm{Ind}\mathcal{C}$ which is equal to the algebra $\mathfrak{u}'$ considered as a $\dot{\mathfrak{u}}$-module by means of the adjoint action.

In the notations of 10.7, let us consider an object

$$\mathbf{ad}_{\mathcal{MS}} := \oplus_{\mu\in\Delta_\ell}\langle L(\mu)\otimes L(\mu)^*\rangle_{\mathcal{MS}} \in \mathcal{MS},$$

cf. [BFM] 4.5.3.

12.6. **Theorem.** *We have a canonical isomorphism* $\langle\mathbf{ad}\rangle_{\mathcal{MS}} = \mathbf{ad}_{\mathcal{MS}}$. $\Box$

13. Regular sheaf

13.1. **Degeneration of quadrics.** The construction below is taken from [KL]II 15.2. Let us consider the quadric $Q \subset \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{A}^1$ given by the equation $uv = t$ where (u, v, t) are coordinates in the triple product. Let $f : Q \longrightarrow \mathbb{A}^1$ be the projection to the third coordinate; for $t \in \mathbb{A}^1$ denote $Q_t := f^{-1}(t)$. For $t \neq 0$, Q_t is isomorphic to $\mathbb{P}^1$; the fiber Q_0 is a union of two projective lines clutched at a point: $Q_0 = Q_u \cup Q_v$ where Q_u (resp. Q_v) is an irreducible component given (in Q_0) by the equation $v = 0$ (resp. $u = 0$) and is isomorphic to $\mathbb{P}^1$; their intersection being a point. We set $'Q = f^{-1}(\mathbb{A}^1 - \{0\})$.

We have two sections $x_1, x_2 : \mathbb{A}^1 \longrightarrow Q$ given by $x_1(t) = (\infty, 0, t)$, $x_2(t) = (0, \infty, t)$. Consider two "coordinate charts" at these points: the maps $\phi_1, \phi_2 : \mathbb{P}^1\times\mathbb{A}^1 \longrightarrow Q$ given by

$$\phi_1(z,t) = (\frac{tz}{z-1}, \frac{z-1}{z}, t);\ \phi_2(z,t) = (\frac{z-1}{z}, \frac{tz}{z-1}, t).$$

This defines a map

(a) $\phi : \mathbb{A}^1 - \{0\} \longrightarrow \tilde{P}(2)$,

in the notations of 7.6.

13.2. For $\nu \in Y^+$, let us consider the corresponding (relative over $\mathbb{A}^1$) configuration scheme $f^\nu : Q^\nu_{/\mathbb{A}^1} \longrightarrow \mathbb{A}^1$. For the brevity we will omit the subscript $_{/\mathbb{A}^1}$ indicating that we are dealing with the relative version of configuration spaces. We denote by $Q^{\nu\bullet}$ (resp. $Q^{\nu\circ}$) the subspace of configurations with the points distinct from x_1, x_2 (resp. also pairwise distinct). We set $'Q^{\nu\circ} = Q^{\nu\circ}|_{\mathbb{A}^1-\{0\}}$, etc.

The map ϕ above, composed with the canonical projection $\tilde{P}(2) \longrightarrow P(2)$, induces the maps

$$'Q^{\nu\circ} \longrightarrow P^\nu(2)\ (\nu \in Y^+)$$

(in the notations of 8.1). For $\nu \in Y^+$ and $\mu_1, \mu_2 \in X_\ell$ such that $\mu_1 + \mu_2 - \nu = 2\rho_\ell$, let $\mathcal{I}^\nu_{\mu_1,\mu_2}$ denote the local system over $'Q^{\nu\circ}$ which is the inverse image of the local system $\mathcal{H}^\nu_{\mu_1,\mu_2}$ over $P^\nu(2)^\circ$. Let $\mathcal{I}^{\nu\bullet}_{\mu_1,\mu_2}$ denote the perverse sheaf over $'Q^{\nu\bullet}(\mathbb{C})$ which is the middle extension of $\mathcal{I}^\nu_{\mu_1,\mu_2}[\dim Q^\nu]$.

13.3. Let us take the nearby cycles and get a perverse sheaf $\Psi_{f^\nu}(\mathcal{I}^{\nu\bullet}_{\mu_1,\mu_2})$ over $Q_0^{\nu\bullet}(\mathbb{C})$. Let us consider the space $Q_0^{\nu\bullet}$ more attentively. This is a reducible scheme which is a union

$$Q_0^{\nu\bullet} = \bigcup_{\nu_1+\nu_2=\nu} \mathbb{A}^{\nu_1} \times \mathbb{A}^{\nu_2},$$

the component $\mathbb{A}^{\nu_1} \times \mathbb{A}^{\nu_2}$ corresponding to configurations where ν_1 (resp. ν_2) points are running on the affine line $Q_u - x_1(0)$ (resp. $Q_v - x_2(0)$). Here we identify these affine lines with a "standard" one using the coordinates u and v respectively. Using this decomposition we can define a closed embedding

$$i_\nu : Q_0^{\nu\bullet} \hookrightarrow \mathbb{A}^\nu \times \mathbb{A}^\nu$$

whose restriction to a component $\mathbb{A}^{\nu_1} \times \mathbb{A}^{\nu_2}$ sends a configuration as above, to the configuration where all remaining points are equal to zero. Let us define a perverse sheaf

$$\mathcal{R}^{\nu,\nu}_{\mu_1,\mu_2} = i_{\nu *}\Psi_{f^\nu}(\mathcal{I}^{\nu\bullet}_{\mu_2,\mu_1}) \in \mathcal{M}(\mathbb{A}^\nu(\mathbb{C}) \times \mathbb{A}^\nu(\mathbb{C}); \mathcal{S})$$

Let us consider the collection of sheaves $\{\mathcal{R}^{\nu,\nu}_{\mu_1,\mu_2} |\ \mu_1,\mu_2 \in X_\ell, \nu \in Y^+, \mu_1+\mu_2-\nu = 2\rho_\ell\}$. One can complete this collection to an object $\mathcal{R}$ of the category $\mathrm{Ind}(\mathcal{FS}\otimes\mathcal{FS})$ where $\mathcal{FS}\otimes\mathcal{FS}$ is understood as a category of finite factorizable sheaves corresponding to the *square* of our initial Cartan datum, i.e. $I \coprod I$, etc. For a precise construction, see [BFS].

13.4. **Theorem.** *We have* $\Phi(\mathcal{R}) = \mathbf{R}$. $\square$

Part III. Modular.

Almost all the results of this part are due to R.Bezrukavnikov.

14. Heisenberg local systems

In this section we sketch a construction of certain remarkable cohesive local systems on arbitrary smooth families of compact smooth curves, to be called the *Heisenberg local systems.*

In the definition and construction of local sustems below we will have to assume that our base field k contains roots of unity of sufficiently high degree; the characteristic of k is assumed to be prime to this degree.

14.1. From now on until 14.11 we fix a smooth proper morphism $f : C \longrightarrow S$ of relative dimension 1, S being a smooth connected scheme over $\mathbb{C}$. For $s \in S$, we denote $C_s := f^{-1}(s)$. Let g be the genus of fibres of f.

Let S_λ denote the total space of the determinant line bundle $\lambda_{C/S} = \det Rf_*\Omega^1_{C/S}$ without the zero section. For any object (?) over S (e.g., a scheme over S, a sheaf over a scheme over S, etc.), we will denote by $(?)_\lambda$ its base change under $S_\lambda \longrightarrow S$.

Below, if we speak about a scheme as a topological (analytic) space, we mean its set of $\mathbb{C}$-points with the usual topology (resp. analytic structure).

14.2. We will use the relative versions of configuration spaces; to indicate this, we will use the subscript $_{/S}$. Thus, if J is a finite set, $C^J_{/S}$ will denote the J-fold fibered product of C with itself over S, etc.

Let $C(J)_{/S}$ denote the subscheme of the J-fold cartesian power of the relative tangent bundle $T_{C/S}$ consisting of J-tuples $\{x_j, \tau_j\}$ where $x_j \in C$ and $\tau_j \neq 0$ in $T_{C/S,x}$, the points x_j being pairwise distinct. Let $\tilde{C}(J)_{/S}$ denote the space of J-tuples of holomorphic embeddings $\phi_j : D \times S \longrightarrow C$ over S with disjoint images; we have the 1-jet maps $\tilde{C}(J)_{/S} \longrightarrow C(J)_{/S}$.

An epimorphism $\rho : K \longrightarrow J$ induces the maps

$$m_{C/S}(\rho) : \prod_J \tilde{D}(K_j) \times \tilde{C}(J)_{/S} \longrightarrow \tilde{C}(J)_{/S}$$

and

$$\bar{m}_{C/S}(\rho) : \prod_J D(K_j) \times \tilde{C}(J)_{/S} \longrightarrow C(K)_{/S}$$

which satisfy the compatibilities as in 7.6.

14.3. We extend the function n to X_ℓ (see 12.1) by $n(\mu)=\frac{1}{2}\mu\cdot\mu-\mu\cdot\rho_\ell$ ($\mu\in X_\ell$). We will denote by $\mathcal{J}$ the X_ℓ-coloured local system over the operad of disks $\mathcal{D}$ which is defined exactly as in 7.5, with X replaced by X_ℓ.

14.4. A *cohesive local system* $\mathcal{H}$ *of level* $\mu\in X_\ell$ *over* C/S is a collection of local systems $\mathcal{H}(\pi)$ over the spaces $C(J)_{/S;\lambda}$ given for every map $\pi: J\longrightarrow X_\ell$ of level μ (note the base change to S_λ!), together with the factorization isomorphisms

$$\phi_C(\rho): m_{C/S}(\rho)^*\mathcal{H}(\pi)\stackrel{\sim}{\longrightarrow}\boxed{\times}_J\mathcal{J}(\pi_j)\boxtimes\mathcal{H}(\rho_*\pi).$$

Here we have denoted by the same letter $\mathcal{H}(\pi)$ the lifting of $\mathcal{H}(\pi)$ to $\tilde{C}_{/S;\lambda}$. The factorization isomorphisms must satisfy the obvious analogs of properties 7.7 (a), (b).

14.5. Now we will sketch a construction of certain cohesive local system over C/S of level $(2-2g)\rho_\ell$. For alternative beautiful constructions of $\mathcal{H}$, see [BP].

To simplify the exposition we will assume below that $g\geq 2$ (the construction for $g\leq 1$ needs some modification, and we omit it here, see [BFS]). Let us consider the group scheme $\mathrm{Pic}(C/S)\otimes X_\ell$ over S. Here $\mathrm{Pic}(C/S)$ is the relative Picard scheme. The group of connected components $\pi_0(\mathrm{Pic}(C/S)\otimes X_\ell)$ is equal to X_ℓ. Let us denote by $\mathcal{J}ac$ the connected component corresponding to the element $(2-2g)\rho_\ell$; this is an abelian scheme over S, due to the existence of the section $S\longrightarrow\mathcal{J}ac$ defined by $\Omega^1_{C/S}\otimes(-\rho_\ell)$.

For a scheme S' over S, let $H_1(S'/S)$ denote the local system of the first relative integral homology groups over S. We have $H_1(\mathcal{J}ac/S)=H_1(C/S)\otimes X_\ell$. We will denote by ω the polarization of $\mathcal{J}ac$ (i.e. the skew symmetric form on the latter local system) equal to the tensor product of the standard form on $H_1(C/S)$ and the form $(\mu_1,\mu_2)\mapsto\frac{\mathrm{d}^g_\ell}{\ell}\mu_1\cdot\mu_2$ on X_ℓ. Note that the assumption $g\geq 2$ implies that $\frac{\mathrm{d}^g_\ell}{\ell}\mu_1\cdot\mu_2\in\mathbb{Z}$ for any $\mu_1,\mu_2\in X_\ell$. Since the latter form is positive definite, ω is relatively ample (i.e. defines a relatively ample invertible sheaf on $\mathcal{J}ac$).

14.6. Let $\alpha=\sum n_\mu\cdot\mu\in\mathbb{N}[X_\ell]$; set $\mathrm{Supp}(\alpha)=\{\mu|\ n_\mu\neq 0\}$. Let us say that α is *admissible* if $\sum n_\mu\mu=(2-2g)\rho_\ell$. Let us denote by

$$\mathrm{aj}_\alpha: C^\alpha_{/S}\longrightarrow\mathrm{Pic}(C/S)\otimes X_\ell$$

the Abel-Jacobi map sending $\sum\mu\cdot x_\mu$ to $\sum x_\mu\otimes\mu$. If α is admissible then the map aj_α lands in $\mathcal{J}ac$.

Let D^α denote the following relative divisor on $C^\alpha_{/S}$

$$D^\alpha=\frac{\mathrm{d}^g_\ell}{\ell}\Big(\sum_{\mu\neq\nu}\mu\cdot\nu\Delta_{\mu\nu}+\frac{1}{2}\sum_\mu\mu\cdot\mu\Delta_{\mu\mu}\Big).$$

Here $\Delta_{\mu\nu}$ ($\mu,\nu\in\mathrm{Supp}(\alpha)$) denotes the corresponding diagonal in $C^\alpha_{/S}$. Note that all the multiplicities are integers.

Let $\pi : J \longrightarrow X_\ell$ be an unfolding of α. We will denote by D^π the pull-back of D^α to $C^J_{/S}$. Let us introduce the following line bundles

$$\mathcal{L}(\pi) = \otimes_{j\in J}\mathcal{T}_j^{\otimes\frac{\mathrm{d}^g_\ell}{\ell}n(\pi(j))} \otimes \mathcal{O}(D^\pi)$$

on $C^J_{/S}$, and

$$\mathcal{L}_\alpha = \mathcal{L}(\pi)/\Sigma_\pi$$

on $C^\alpha_{/S}$ (the action of Σ_π is an obvious one). Here $\mathcal{T}_j$ denotes the relative tangent line bundle on $C^J_{/S}$ in the direction j. Note that the numbers $\frac{\mathrm{d}^g_\ell}{\ell}n(\mu)$ $(\mu \in X_\ell)$ are integers.

14.7. **Proposition.** *There exists a unique line bundle $\mathcal{L}$ on $\mathcal{J}ac$ such that for each admissible α, we have $\mathcal{L}_\alpha = \mathrm{aj}^*_\alpha(\mathcal{L})$. The first Chern class $c_1(\mathcal{L}) = -[\omega]$.* □

14.8. In the sequel if $\mathcal{L}_0$ is a line bundle, let $\overset{\bullet}{\mathcal{L}}_0$ denote the total space its with the zero section removed.

The next step is the construction of a certain local system $\mathfrak{H}$ over $\overset{\bullet}{\mathcal{L}}_\lambda$. Its dimension is equal to d^g_ℓ and the monodromy around the zero section of $\mathcal{L}$ (resp. of the determinant bundle) is equal to $\zeta^{-2\ell/\mathrm{d}^g_\ell}$ (resp. $(-1)^{\mathrm{rk}(X)}\zeta^{-12\rho_\ell\cdot\rho_\ell}$). The construction of $\mathfrak{H}$ is outlined below.

The previous construction assigns to a triple

(a lattice Λ, a symmetric bilinear form $(\ ,\ ):\ \Lambda\times\Lambda \longrightarrow \mathbb{Z}$, $\nu \in \Lambda$)

an abelian scheme $\mathcal{J}ac_\Lambda := (\mathrm{Pic}(C/S)\otimes\Lambda)_{(2g-2)\nu}$ over S, together with a line bundle $\mathcal{L}_\Lambda$ on it (in the definition of $\mathcal{L}_\Lambda$ one should use the function $n_\nu(\mu) = \frac{1}{2}\mu\cdot\mu + \mu\cdot\nu$). We considered the case $\Lambda = X_\ell,\ (\mu_1,\mu_2) = \frac{\mathrm{d}^g_\ell}{\ell}\mu_1\cdot\mu_2,\ \nu = -\rho_\ell$.

Now let us apply this construction to the lattice $\Lambda = X_\ell\oplus Y_\ell$, the bilinear form $((\mu_1,\nu_1),(\mu_2,\nu_2)) = -\frac{1}{\ell}(\nu_1\cdot\mu_2 + \nu_2\cdot\mu_1 + \nu_1\cdot\nu_2)$ and $\nu = (-\rho_\ell, 0)$. The first projection $\Lambda \longrightarrow X_\ell$ induces the morphism

$$p:\ \mathcal{J}ac_\Lambda \longrightarrow \mathcal{J}ac$$

the fibers of p are abelian varieties $\mathcal{J}ac(C_s)\otimes Y_\ell$ $(s\in S)$.

14.9. **Theorem.** (i) *The line bundle $\mathcal{L}_\Lambda$ is relatively ample with respect to p. The direct image $\mathcal{E} := p_*\mathcal{L}_\Lambda$ is a locally free sheaf of rank d^g_ℓ.*

(ii) *We have an isomorphism*

$$\det(\mathcal{E}) = \mathcal{L}\otimes\lambda^{\mathrm{d}^g_\ell(-\frac{1}{2}\mathrm{rk}(X_\ell)+6\frac{\rho_\ell\cdot\rho_\ell}{\ell})}.$$

□

Here λ denotes the pull-back of the determinant bundle $\lambda_{C/S}$ to $\mathcal{J}ac$.

14.10. Let us assume for a moment that $k = \mathbb{C}$ and $\zeta = \exp(-\frac{\pi\sqrt{-1}}{\ell})$. By the result of Beilinson-Kazhdan, [BK] 4.2, the vector bundle $\mathcal{E}$ carries a canonical flat projective connection. By *loc. cit.* 2.5, its lifting to $\det(\mathcal{E})^\bullet$ carries a flat connection with the scalar monodromy around the zero section equal to $\exp(\frac{2\pi\sqrt{-1}}{\mathrm{d}^g_\ell})$. We have an obvious map

$$m : \overset{\bullet}{\mathcal{L}}_\lambda \longrightarrow \mathcal{L} \otimes \lambda^{\mathrm{d}^g_\ell(-\frac{1}{2}\mathrm{rk}(X_\ell)+6\frac{\rho_\ell\cdot\rho_\ell}{\ell})}.$$

By definition, $\mathfrak{H}$ is the local system of horizontal sections of the pull-back of $\mathcal{E}$ to $\overset{\bullet}{\mathcal{L}}_\lambda$. The claim about its monodromies follows from part (ii) of the previous theorem.

This completes the construction of $\mathfrak{H}$ for $k = \mathbb{C}$ and $\zeta = \exp(-\frac{\pi\sqrt{-1}}{\ell})$. The case of arbitrary k (of sufficiently large characteristic) and ζ follows from this one.

14.11. Let us consider an obvious map $q : C(J)_{/S;\lambda} \longrightarrow C^J_{/S;\lambda}$. The pull-back $q^*\mathcal{L}(\pi)$ has a canonical non-zero section s. Let $\tilde{\mathcal{H}}(\pi)$ be the pull-back of the local system $\mathfrak{H}$ to $q^*\mathcal{L}(\pi)$. By definition, we set $\mathcal{H}(\pi) = s^*\tilde{\mathcal{H}}(\pi)$. For the construction of the factorization isomorphisms, see [BFS].

14.12. Let $\mathfrak{g}$ be the simple Lie algebra connected with our Cartan datum. Assume that $\zeta = \exp(\frac{\pi\sqrt{-1}}{d\kappa})$ for some positive integer κ, cf. 10.6 (d is defined in 2.1).

We have $12\rho_\ell \cdot \rho_\ell \equiv 12\rho\cdot\rho \bmod l$, and $\mathrm{rk}(X) \equiv \dim\mathfrak{g} \bmod 2$. By the strange formula of Freudenthal-de Vries, we have $12\rho\cdot\rho = dh\dim\mathfrak{g}$ where h is the dual Coxeter number of our Cartan datum. It follows that the monodromy of $\mathcal{H}$ around the zero section of the determinant line bundle is equal to $\exp(\pi\sqrt{-1}\frac{(\kappa-h)\dim\mathfrak{g}}{\kappa})$. This number coincides with the multiplicative central charge of the conformal field theory associated with the affine Lie algebra $\hat{\mathfrak{g}}$ at level κ (see [BFM] 4.4.1, 6.1.1, 2.1.3, [TUY] 1.2.2), cf 16.2 below.

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14.13. Let us define a category $\mathcal{S}ew$ as follows (cf. [BFM] 4.3.2). Its object A is a finite set $\overline{A}$ together with a collection $N_A = \{n\}$ of non-intersecting two-element subsets $n \subset \overline{A}$. Given such an object, we set $A^1 = \bigcup_{N_A} n,\ A^0 = \overline{A} - A^1$. A morphism $f : A \longrightarrow B$ is an embedding $i_f : \overline{B} \hookrightarrow \overline{A}$ and a collection N_f of non-intersecting two-element subsets of $\overline{B} - \overline{A}$ such that $N_A = N_B \coprod N_f$. The composition of morphisms is obvious. ($\mathcal{S}ew$ coincides with the category $Sets^\sharp/\emptyset$, in the notations of [BFM] 4.3.2.)

For $A \in \mathcal{S}ew$, let us call an *A-curve* a data $(C, \{x_a, \tau_a\}_{A^0})$ where C is a smooth proper (possibly disconnected) complex curve, $\{x_a, \tau_a\}_{A^0}$ is an A^0-tuple of distinct points $x_a \in C$ together with non-zero tangent vectors τ_a at them. For such a curve, let $\overline{C}_A$ denote the curve obtained from C by clutching pairwise the points $x_{a'}$ with $x_{a''}$ ($n = \{a', a''\}$) for all sets $n \in N_A$. Thus, the set N_A is in the bijection with the set of nodes of the curve $\overline{C}_A$ ($\overline{C}_A = C$ if $N_A = \emptyset$).

Let us call an *enhanced graph* a pair $\Gamma = (\overline{\Gamma}, \mathbf{g})$ here $\overline{\Gamma}$ is a non-oriented graph and $\mathbf{g} = \{g_v\}_{v\in \mathrm{Vert}(\overline{\Gamma})}$ is a $\mathbb{N}$- valued 0-chain of $\overline{\Gamma}$. Here $\mathrm{Vert}(\overline{\Gamma})$ denotes the set of vertices of $\overline{\Gamma}$. Let us assign to a curve $\overline{C}_A$ an enhanced graph $\Gamma(\overline{C}_A) = (\overline{\Gamma}(\overline{C}_A), \mathbf{g}(\overline{C}_A))$. By definition, $\overline{\Gamma}(\overline{C}_A)$ is a graph with $\mathrm{Vert}(\overline{\Gamma}(\overline{C}_A)) = \pi_0(C) = \{$ the set of irreducible components of $\overline{C}_A\}$ and the set of edges $\mathrm{Edge}(\overline{\Gamma}(\overline{C}_A)) = N_A$, an edge $n = \{a', a''\}$ connecting the vertices corresponding to the components of the points $x_{a'}, x_{a''}$. For $v \in \pi_0(C)$, $g(\overline{C}_A)_v$ is equal to the genus of the corresponding component $C_v \subset C$.

14.14. Let $\mathcal{M}_A$ denote the moduli stack of A-curves $(C, \dots)$ such that the curve $\overline{C}_A$ is stable in the sense of [DM] (in particular connected). The stack $\mathcal{M}_A$ is smooth; we have $\mathcal{M}_A = \coprod_{g\geq 0} \mathcal{M}_{A,g}$ where $\mathcal{M}_{A,g}$ is a substack of A-curves C with $\overline{C}_A$ having genus (i.e. $\dim H^1(\overline{C}_A, \mathcal{O}_{\overline{C}_A})$) equal to g. In turn, we have the decomposition into connected components

$$\mathcal{M}_{A,g} = \coprod_{\Gamma, \vec{A}^0} \mathcal{M}_{\vec{A}^0, g, \Gamma}$$

where $\mathcal{M}_{\vec{A}^0,g,\Gamma}$ is the stack of A-curves $(C, \{x_a, \tau_a\})$ as above, with $\Gamma(\overline{C}_A) = \Gamma$, $\vec{A}^0 = \{A^0_v\}_{v\in\mathrm{Vert}(\overline{\Gamma})}$, $A^0 = \coprod\ A^0_v$, such that x_a lives on the connected component C_v for $a \in A^0_v$.

We denote by $\eta : C_{A,g} \longrightarrow \mathcal{M}_{A,g}$ (resp. $\bar{\eta} : \overline{C}_{A,g} \longrightarrow \mathcal{M}_{A,g}$) the universal smooth curve (resp. stable surve). For $\nu \in Y^+$, we have the corresponding *relative* configuration spaces $C^\nu_{A,g}, C^{\nu\circ}_{A,g}, \overline{C}^\nu_{A,g}$. For brevity we omit the relativeness subscript $/\mathcal{M}_{A,g}$ from these notations. The notation $C^\nu_{\vec{A}^0,g,\Gamma}$ etc., will mean the restriction of these configuration spaces to the component $\mathcal{M}_{\vec{A}^0,g,\Gamma}$.

Let $\mathcal{M}_g$ be the moduli stack of smooth connected curves of genus g, and $\overline{\mathcal{M}}_g$ be its Grothendieck-Deligne-Mumford-Knudsen compactification, i.e. the moduli stack of stable curves of genus g. Let $\bar{\eta} : \overline{C}_g \longrightarrow \overline{\mathcal{M}}_g$ be the universal stable curve; let $\bar{\lambda}_g = \det R\bar{\eta}_*(\omega_{\overline{C}_g/\overline{\mathcal{M}}_g})$ be the determinant line bundle; let $\overline{\mathcal{M}}_{g;\lambda} \longrightarrow \overline{\mathcal{M}}_g$ be its total space with the zero section removed.

We have obvious maps $\mathcal{M}_{A,g} \longrightarrow \overline{\mathcal{M}}_g$. Let the complementary subscript $(\cdot)_\lambda$ denote the base change of all the above objects under $\overline{\mathcal{M}}_{g;\lambda} \longrightarrow \overline{\mathcal{M}}_g$.

14.15. Let us consider the configuration space $C^{\nu\circ}_{A,g}$; it is the moduli stack of ν distinct points running on A-curves $(C, \{x_a, \tau_a\})$ and not equal to the marked points x_a. This stack decomposes into connected components as follows:

$$C^{\nu\circ}_{A,g} = \coprod_{\Gamma, \vec{A}^0, \vec{\nu}} C^{\vec{\nu}}_{\vec{A}^0,g,\Gamma}$$

where $\vec{\nu} = \{\nu_v\}_{v\in\mathrm{Vert}(\Gamma)}$ and $C^{\vec{\nu}}_{\vec{A}^0,g,\Gamma}$ being the moduli stack of objects as above, with $\Gamma(\overline{C}_A) = \Gamma$ and ν_v points running on the component C_v. The decomposition is taken over appropriate graphs Γ, decompositions $A^0 = \coprod A^0_v$ and the tuples $\vec{\nu}$ with $\sum \nu_v = \nu$.

Let us call an A_0-tuple $\vec{\mu}=\{\mu_a\}\in X_\ell^{A^0}$ (g,ν)-*good* if

(a) $\sum_{a\in A^0}\ \mu_a-\nu\equiv(2-2g)\rho_\ell \bmod Y_\ell$.

Given such a tuple, we are going to define certain local system $\mathcal{H}^{\nu}_{\vec{\mu};A,g}$ over $C^{\nu\circ}_{A,g;\lambda}$. Let us describe its restriction $\mathcal{H}^{\vec{\nu}}_{\vec{\mu};\vec{A}^0,g,\Gamma}$ to a connected component $C^{\vec{\nu}}_{\vec{A}^0,g,\Gamma;\lambda}$.

Let Γ' be the first subdivision of $\overline{\Gamma}$. We have $\mathrm{Vert}(\Gamma')=\mathrm{Vert}(\overline{\Gamma})\coprod\mathrm{Edge}(\overline{\Gamma})=\pi_0(C)\coprod N_A$. The edges of Γ' are indexed by the pairs (n,a) where $n\in N_A, a\in n$, the corresponding edge $e_{n,a}$ having the ends a and n. Let us define an orientation of Γ' by the requierement that a is the beginning of $e_{n,a}$. Consider the chain complex

$$C_1(\Gamma';X_\ell/Y_\ell)\xrightarrow{d}C_0(\Gamma';X_\ell/Y_\ell).$$

Let us define a 0-chain $c=c^{\vec{\nu}}_{\vec{\mu}}\in C_0(\Gamma';X_\ell/Y_\ell)$ by

$$c(v)=\sum_{a\in A^0_v}\ \mu_a+(2g_v-2)\rho_\ell-\nu_v\ (v\in\pi_0(C));\ c(n)=2\rho_\ell\ (n\in N_A).$$

The goodness assumption (a) ensures that c is a boundary. By definition,

$$\mathcal{H}^{\vec{\nu}}_{\vec{\mu};\vec{A}^0,g,\Gamma}=\oplus_{\chi:\ d\chi=c}\ \mathcal{H}_\chi$$

Note that the set $\{\chi|\ d\chi=c\}$ is a torsor over the group $H_1(\Gamma';X_\ell/Y_\ell)=H_1(\Gamma;X_\ell/Y_\ell)$.

The local system $\mathcal{H}_\chi$ is defined below, in 14.18, after a little notational preparation.

14.16. Given two finite sets J,K, let $C(J;K)_g$ denote the moduli stack of objects $(C,\{x_j\},\{y_k,\tau_k\})$. Here C is a smooth proper connected curve of genus g, $\{x_j\}$ is a J-tuple of distinct points $x_j\in C$ and $\{y_k,\tau_k\}$ is a K-tuple of distinct points $y_k\in C$ together with non-zero tangent vectors $\tau_k\in T_{y_k}C$. We suppose that $y_k\neq x_j$ for all k,j.

We set $C(J)_g:=C(\emptyset;J)_g$. We have the forgetful maps $C(J\coprod K)_g\longrightarrow C(J;K)_g$.

The construction of 14.5 — 14.11 defines the Heisenberg system $\mathcal{H}(\pi)$ over the smooth stack $C(J)_{g;\lambda}$ for each $\pi:J\longrightarrow X_\ell$.

Given $\nu\in Y^+$, choose an unfolding of ν, $\pi:J\longrightarrow I$, and set $C^\nu(K)^\circ_g:=C(J;K)/\Sigma_\pi$. Given a K-tuple $\vec{\mu}=\{\mu_k\}\in X_\ell^K$, define a map $\tilde{\pi}:J\coprod K\longrightarrow X_\ell$ by $\tilde{\pi}(j)=-\pi(j)\in -I\subset X_\ell\ (j\in J)$, $\tilde{\pi}(k)=\mu_k\ (k\in K)$. The local system $\mathcal{H}(\tilde{\pi})$ over $C(J\coprod K)_{g;\lambda}$ descends to $C(J;K)_{g;\lambda}$ since $\zeta^{2n(-i)}=1$, and then to $C^\nu(K)^\circ_{g;\lambda}$, by Σ_π-equivariance; denote the latter local system by $\tilde{\mathcal{H}}^\nu_{\vec{\mu}}$, and set $\mathcal{H}^\nu_{\vec{\mu}}=\tilde{\mathcal{H}}^\nu_{\vec{\mu}}\otimes\mathcal{S}ign^\nu$, cf. 8.3.

14.17. **Lemma.** *If $\vec{\mu}\equiv\vec{\mu}'$ mod Y_ℓ^K then we have canonical isomorphisms $\mathcal{H}^\nu_{\vec{\mu}}=\mathcal{H}^\nu_{\vec{\mu}'}$.* $\square$

Therefore, it makes sense to speak about $\mathcal{H}^\nu_{\vec{\mu}}$ for $\vec{\mu}\in(X_\ell/Y_\ell)^K$.

14.18. Let us return to the situation at the end of 14.15. We have $\Gamma = (\overline{\Gamma}, \{g_v\}_{v\in\mathrm{Vert}(\overline{\Gamma})})$. Recall that $A^0 = \coprod A^0_v$; we have also $A^1 = \coprod A^1_v$ where $A^1_v := \{a \in A^1 |\ x_a \in C_v\}$ ($v \in \mathrm{Vert}(\overline{\Gamma})$). Set $\overline{A}_v = A^0_v \coprod A^1_v$, so that $\overline{A} = \coprod \overline{A}_v$. We have an obvious map

(a) $\mathcal{M}_{\vec{A}^0,g,\Gamma} \longrightarrow \prod_{\mathrm{Vert}(\overline{\Gamma})} C(\overline{A}_v)_{g_v}$,

and a map

(b) $C^{\vec{\nu}}_{\vec{A}^0,g,\Gamma} \longrightarrow \prod_{\mathrm{Vert}(\overline{\Gamma})} C^{\nu_v}(\overline{A}_v)^\circ_{g_v}$

over (a). For each v, define an $\overline{A}_v$-tuple $\vec{\mu}(\chi; v)$ equal to μ_a at $a \in A^0_v$ and $\chi(e_{a,n})$ at $a \in n \subset A^1$. By definition, the local system $\mathcal{H}_\chi$ over $C^{\vec{\nu}}_{\vec{A}^0,g,\Gamma;\lambda}$ is the inverse image of the product

$\boxed{\times}_{\mathrm{Vert}(\overline{\Gamma})} \mathcal{H}^{\nu_v}_{\vec{\mu}(\chi;v)}$

under the map (b) (pulled back to the determinant bundle).

This completes the definition of the local systems $\mathcal{H}^\nu_{\vec{\mu};A,g}$. They have a remarkable compatibility property (when the object A varies) which we are going to describe below, see theorem 14.23.

14.19. Let $\tilde{\mathcal{T}}^\nu_{A,g}$ denote the fundamental groupoid $\pi(C^{\nu\circ}_{A,g;\lambda})$. We are going to show that these groupoids form a cofibered category over $\mathcal{S}ew$.

14.20. A morphism $f : A \longrightarrow B$ in $\mathcal{S}ew$ is called a *sewing* (resp. *deleting*) if $A^0 = B^0$ (resp. $N_f = \emptyset$). A sewing f with $\mathrm{card}(N_f) = 1$ is called *simple*. Each morphism is a composition of a sewing and a deleting; each sewing is a composition of simple ones.

(a) Let $f : A \longrightarrow B$ be a simple sewing. We have canonical morphisms

$$\wp_f : \mathcal{M}_{A,g;\lambda} \longrightarrow \overset{\bullet}{T}_{\partial\overline{\mathcal{M}}_{B,g;\lambda}} \overline{\mathcal{M}}_{B,g;\lambda}$$

and

$$\wp^{(\nu\circ)}_f : C^{\nu\circ}_{A,g;\lambda} \longrightarrow \overset{\bullet}{T}_{\partial\overline{C^{\nu\circ}_{B,g;\lambda}}} \overline{C^{\nu\circ}_{B,g;\lambda}},$$

over $\wp_f$, cf. [BFM] 4.3.1. Here $\overline{\mathcal{M}}_{B,g}$ (resp. $\overline{C^{\nu\circ}_{B,g}}$) denotes the Grothendieck-Deligne-Mumford-Knudsen compactification of $\mathcal{M}_{B,g}$ (resp. of $C^{\nu\circ}_{B,g}$) and $\partial\overline{\mathcal{M}}_{B,g}$ (resp. $\partial\overline{C^{\nu\circ}_{B,g}}$) denotes the smooth locus of the boundary $\overline{\mathcal{M}}_{B,g} - \mathcal{M}_{B,g}$ (resp. $\overline{C^{\nu\circ}_{B,g}} - \overline{C}^{\nu}_{B,g}$). The subscript $_\lambda$ indicates the base change to the determinant bundle, as before.

Composing the specialization with the inverse image under $\wp^{(\nu\circ)}_f$, we get the canonical map $f_* : \tilde{T}^\nu_{A,g} \longrightarrow \tilde{T}^\nu_{B,g}$.

(b) Let $f : A \longrightarrow B$ be a deleting. It induces the obvious morphisms (denoted by the same letter)

$$f : \mathcal{M}_{A,g;\lambda} \longrightarrow \mathcal{M}_{B,g;\lambda}$$

and

$$f: C^{\nu\circ}_{A,g;\lambda} \longrightarrow C^{\nu\circ}_{B,g;\lambda}.$$

The last map induces $f_*: \tilde{T}^\nu_{A,g} \longrightarrow \tilde{T}^\nu_{B,g}$.

Combining the constructions (a) and (b) above, we get a category $\tilde{T}^\nu_g$ cofibered in groupoids over $\mathcal{S}ew$, with fibers $\tilde{T}^\nu_{A,g}$.

14.21. Let $\mathcal{R}ep^\nu_{c;A,g}$ be the category of finite dimensional representations of $\tilde{T}^\nu_{A,g}$ (over k) with the monodromy $c \in k^*$ around the zero section of the determinant bundle. The previous construction shows that these categories form a fibered category $\mathcal{R}ep^\nu_{c;g}$ over $\mathcal{S}ew$.

14.22. For $A \in \mathcal{S}ew$, let us call an A^0-tuple $\vec{\mu} = \{\mu_a\} \in X_\ell^{A^0}$ *good* if $\sum_{A^0}\ \mu_a \in Y$.

If $f: B \longrightarrow A$ is a morphism, define a B^0-tuple $f^*\vec{\mu} = \{\mu'_b\}$ by $\mu'_b = \mu_{i_f^{-1}(b)}$ if $b \in i_f(A^0)$, and 0 otherwise. Obviously, $\sum_{B^0}\ \mu'_b = \sum_{A^0}\ \mu_a$.

Given a good $\vec{\mu}$, let us pick an element $\nu \in Y^+$ such that $\nu \equiv \sum \mu_a + (2g-2)\rho_\ell \mod Y_\ell$. We can consider the local system $\mathcal{H}^\nu_{\vec{\mu};A,g}$ as an object of $\mathcal{R}ep^\nu_{c;A,g}$ where $c = (-1)^{\mathrm{card}(I)}\zeta^{-12\rho\cdot\rho}$.

14.23. **Theorem.** *For any morphism $f: B \longrightarrow A$ in $\mathcal{S}ew$ and a g-good $\vec{\mu} \in X_\ell^{A^0}$, we have the canonical isomorphism*

$$f^*\mathcal{H}^\nu_{\vec{\mu};A,g} = \mathcal{H}^\nu_{f^*\vec{\mu};B,g}.$$

In other words, the local systems $\mathcal{H}^\nu_{\vec{\mu};B,g}$ define a **cartesian section** *of the fibered category $\mathcal{R}ep^\nu_{c;g}$ over $\mathcal{S}ew/A$. Here $c = (-1)^{\mathrm{card}(I)}\zeta^{-12\rho\cdot\rho}$.* $\square$

15. Fusion structures on $\mathcal{FS}$

15.1. Below we will construct a family of "fusion structures" on the category $\mathcal{FS}$ (and hence, due to the equivalence Φ, on the category $\mathcal{C}$) indexed by $m \in \mathbb{Z}$. We should explain what a fusion structure is. This is done in 15.8 below. We will use a modification of the formalism from [BFM] 4.5.4.

15.2. Recall that we have a regular object $\mathcal{R} \in \mathrm{Ind}(\mathcal{FS}^{\otimes 2})$, cf. Section 13. We have the canonical isomorphism $t(\mathcal{R}) = \mathcal{R}$ where $t: \mathrm{Ind}(\mathcal{FS}^{\otimes 2}) \xrightarrow{\sim} \mathrm{Ind}(\mathcal{FS}^{\otimes 2})$ is the permutation, hence an object $\mathcal{R}_n \in \mathrm{Ind}(\mathcal{FS}^{\otimes 2})$ is well defined for any two-element set n.

For an object $A \in \mathcal{S}ew$, we set $\tilde{A} = A^0 \coprod N_A$. Let us call an *$A$-collection* of factorizable sheaves an $\tilde{A}$-tuple $\{\mathcal{X}_{\tilde{a}}\}_{\tilde{a}\in\tilde{A}}$ where $\mathcal{X}_{\tilde{a}} \in \mathcal{FS}_{c_{\tilde{a}}}$ if $\tilde{a} \in A^0$ and $\mathcal{X}_{\tilde{a}} = \mathcal{R}_{\tilde{a}}$ if $\tilde{a} \in N_A$. We impose the condition that $\sum_{a\in A^0}\ c_a = 0 \in X_\ell/Y$. We will denote such an object $\{\mathcal{X}_a; \mathcal{R}_n\}_A$. It defines an object

$$\otimes_A\ \{\mathcal{X}_a; \mathcal{R}_n\}\ := (\otimes_{a\in A^0}\ \mathcal{X}_a) \otimes (\otimes_{n\in N_A}\ \mathcal{R}_n) \in \mathrm{Ind}(\mathcal{FS}^{\otimes\overline{A}}).$$

If $f : B \longrightarrow A$ is a morphism in $\mathcal{S}ew$, we define a B-collection $f^*\{\mathcal{X}_a; \mathcal{R}_n\}_B = \{\mathcal{Y}_{\tilde{b}}\}_{\tilde{B}}$ by $\mathcal{Y}_{\tilde{b}} = \mathcal{X}_{i_f^{-1}(\tilde{b})}$ for $\tilde{b} \in i_f(A^0)$, $\mathbf{1}$ if $\tilde{b} \in B^0 - i_f(A^0)$ and $\mathcal{R}_{\tilde{b}}$ if $\tilde{b} \in N_B = N_A \cup N_f$.

15.3. Given an A-collection $\{\mathcal{X}_a, \mathcal{R}_n\}_A$ with $\lambda(\mathcal{X}_a) = \mu_a$ such that

(a) $\nu := \sum_{A^0} \mu_a + (2g-2)\rho_\ell \in Y^+$,

one constructs (following the pattern of 8.5) a perverse sheaf $\boxed{\times}^{(\nu)}_{A,g} \{\mathcal{X}_a; \mathcal{R}_n\}$ over $\overline{C}^\nu_{A,g;\lambda}$. It is obtained by planting factorizable sheaves $\mathcal{X}_a$ into the universal sections x_a of the stable curve $\overline{C}_{A,g}$, the regular sheaves $\mathcal{R}_n$ into the nodes n of this curve and pasting them together into one sheaf by the Heisenberg system $\mathcal{H}^\nu_{\vec{\mu};A,g}$.

15.4. Given $A \in \mathcal{S}ew$ and an A-collection $\{\mathcal{X}_a; \mathcal{R}_n\}_A$, choose elements $\mu_a \geq \lambda(\mathcal{X}_a)$ in X_ℓ such that (a) above holds (note that $2\rho_\ell \in Y$). Below ν will denote the element as in (a) above.

Let $\mathcal{X}'_a$ denote the factorizable sheaf isomorphic to $\mathcal{X}_a$ obtained from it by the change of the highest weight from $\lambda(\mathcal{X}_a)$ to μ_a. For each $m \in \mathbb{Z}$, define a local system $\langle \otimes_A \{\mathcal{X}_a; \mathcal{R}_n\}\rangle^{(m)}_g$ over $\mathcal{M}_{A,g;\lambda}$ as follows.

Let $\langle \otimes_A \{\mathcal{X}_a; \mathcal{R}_n\}\rangle^{(m)}_{\vec{A}^0,g,\Gamma}$ denote the restriction of the local system to be defined to the connected component $\mathcal{M}_{\vec{A}^0,g,\Gamma;\lambda}$. By definition,

$$\langle \otimes_A \{\mathcal{X}_a; \mathcal{R}_n\}\rangle^{(m)}_{\vec{A}^0,g,\Gamma} := R^{m-\dim \mathcal{M}_{\vec{A}^0,g,\Gamma;\lambda}} \bar{\eta}^\nu_* (\boxed{\times}^{(\nu)}_{\vec{A}^0,g,\Gamma} \{\mathcal{X}'_a; \mathcal{R}_n\}).$$

Here $\boxed{\times}^{(\nu)}_{\vec{A}^0,g,\Gamma} \{\mathcal{X}'_a; \mathcal{R}_n\}$ denotes the perverse sheaf $\boxed{\times}^{(\nu)}_{A,g} \{\mathcal{X}'_a; \mathcal{R}_n\}$ restricted to the subspace $\overline{C}^\nu_{\vec{A}^0,g,\Gamma;\lambda}$. This definition does not depend on the choice of the elements μ_a.

15.5. Given a morphism $f : B \longrightarrow A$, we define, acting as in 14.19, 14.21, a perverse sheaf $f^*(\boxed{\times}^{(\nu)}_{A,g} \{\mathcal{X}_a; \mathcal{R}_n\})$ over $\overline{C}^\nu_{B,g;\lambda}$ and local systems $f^*\langle \otimes_A \{\mathcal{X}_a; \mathcal{R}_n\}\rangle^{(m)}_g$ over $\mathcal{M}_{B,g;\lambda}$.

15.6. **Theorem.** *In the above notations, we have canonical isomorphisms*

$$f^*(\boxed{\times}^{(\nu)}_{A,g} \{\mathcal{X}_a; \mathcal{R}_n\}) = \boxed{\times}^{(\nu)}_{B,g} f^*\{\mathcal{X}_a; \mathcal{R}_n\}.$$

This is a consequence of Theorem 14.23 above and the definition of the regular sheaf $\mathcal{R}$ as a sheaf of nearby cycles of the braiding local system, 13.3.

15.7. **Corollary.** *We have canonical isomorphisms of local systems*

$$f^*\langle \otimes_A \{\mathcal{X}_a; \mathcal{R}_n\}\rangle^{(m)}_g = \langle \otimes_B f^*\{\mathcal{X}_a; \mathcal{R}_n\}\rangle^{(m)}_g \ (m \in \mathbb{Z}). \ \Box$$

15.8. The previous corollary may be expressed as follows. The various A-collections of factorizable sheaves (resp. categories $\widetilde{\mathcal{R}ep}_{c;A}$ of finite dimensional representations of "Teichmüller groupoids" $\widetilde{\mathcal{T}eich}_A = \pi(\mathcal{M}_{A;\lambda})$ having monodromy c around the zero section of the determinant bundle) define a fibered category $\mathcal{FS}^\sharp$ (resp. $\widetilde{\mathcal{R}ep}_c$) over $\mathcal{S}ew$.

For any $m \in \mathbb{Z}$, the collection of local systems $\langle \otimes_A \{\mathcal{X}_a; \mathcal{R}_n\}\rangle_g^{(m)}$ and the canonical isomorphisms of the previous theorem define a **cartesian functor**

$$\langle\ \rangle^{(m)} : \mathcal{FS}^\sharp \longrightarrow \widetilde{\mathcal{R}ep}_c$$

where $c = (-1)^{\mathrm{card}(I)}\zeta^{-12\rho\cdot\rho}$. We call such a functor a *fusion structure of multiplicative central charge* c on the category $\mathcal{FS}$. The category $\mathcal{FS}$ with this fusion structure will be denoted by $\mathcal{FS}^{(m)}$.

The difference from the definition of a fusion structure given in [BFM] 4.5.4 is that our fibered categories live over $\mathcal{S}ew = \mathcal{S}ets^\sharp/\emptyset$ and not over $\mathcal{S}ets^\sharp$, as in *op. cit.*

15.9. **Example.** For $A \in \mathcal{S}ew$, let us consider an A-curve $P = (\mathbb{P}^1, \{x_a, \tau_a\})$; it defines a geometric point Q of the stack $\mathcal{M}_{A,g}$ where $g = \mathrm{card}(N_A)$ and hence a geometric point $P = (Q, 1)$ of the stack $\mathcal{M}_{A,g;\lambda}$ since the determinant bundle is canonically trivialized at Q.

15.9.1. **Theorem.** *For an A-collection $\{\mathcal{X}_a; \mathcal{R}_n\}$, the stalk of the local system $\langle \otimes_A \mathcal{X}_a\rangle_g^{(m)}$ at a point P is isomorphic to*

$$\mathrm{Tor}^{\mathcal{C}}_{\frac{\infty}{2}-m}(\mathbf{1}_r, (\otimes_{A^0}\ \Phi(\mathcal{X}_a)) \otimes \mathbf{ad}^{\otimes g}).$$

To prove this, one should apply theorem 11.2 and the following remark. Degenerating all nodes into cusps, one can include the nodal curve $\overline{P}_A$ into a one-parameter family whose special fiber is $\mathbb{P}^1$, with $\mathrm{card}(A^0) + g$ marked points. The nearby cycles of the sheaf $\boxed{\times}_A \{\mathcal{X}_a; \mathcal{R}_n\}$ will be the sheaf obtained by the gluing of the sheaves $\mathcal{X}_a$ and g copies of the sheaf $\Phi^{-1}(\mathbf{ad})$ into these marked points.

15.10. Note that an arbitrary A-curve may be degenerated into a curve considered in the previous example. Due to 15.7, this determines the stalks of all our local systems (up to a non canonical isomorphism).

16. Conformal blocks (higher genus)

In this section we assume that $k = \mathbb{C}$.

16.1. Let us make the assumptions of 10.6, 14.12. Consider the full subcategory $\mathcal{MS} \subset \mathcal{C}$. Let us define the *regular object* $\mathbf{R}_{\mathcal{MS}}$ by

$$\mathbf{R}_{\mathcal{MS}} = \oplus_{\mu \in \Delta_\ell} \ (\mathfrak{L}(\mu) \otimes \mathfrak{L}(\mu)^*) \in \mathcal{MS}^{\otimes 2}.$$

As in the previous section, we have a notion of A-collection $\{L_a; \mathbf{R}_{\mathcal{MS}n}\}_A$ ($A \in \mathcal{S}ew, \{L_a\} \in \mathcal{MS}^{A^0}$). The classical fusion structure on $\mathcal{MS}$, [TUY], defines for each A-collection as above, a local system

$$\langle \otimes_A \ \{L_a; \mathbf{R}_{\mathcal{MS}n}\} \rangle_{\mathcal{MS}}$$

on the moduli stack $\mathcal{M}_{A;\lambda}$.

We have $A = (\overline{A}, N_A)$; let us define another object $A' = (\overline{A} \cup \{*\}, N_A)$. We have an obvious deleting $f_A : A' \longrightarrow A$. Given an A-collection as above, define an A'-collection in $\{L_a; L(2\rho_\ell); \mathbf{R}_n\}_{A'}$ in the category $\mathcal{C}$. Using the equivalence Φ, we transfer to $\mathcal{C}$ the fusion structures defined in the previous section on $\mathcal{FS}$; we denote them $\langle\ \rangle_{\mathcal{C}}^{(m)}$.

The following theorem generalizes Theorem 11.5 to higher genus.

16.2. **Theorem.** *For each $A \in \mathcal{S}ew$, the local system $f_A^* \langle \otimes_A \ \{L_a; \mathbf{R}_{\mathcal{MS}n}\} \rangle_{\mathcal{MS}}$ on $\mathcal{M}_{A';\lambda}$ is a canonical subquotient of the local system $\langle \otimes_{A'} \ \{L_a; L(2\rho_\ell); \mathbf{R}_n\} \rangle_{\mathcal{C}}^{(0)}$.* $\square$

16.3. Let us consider the special case $A = (A^0, \emptyset)$; for an A-collection $\{L_a\}_{A^0}$ in $\mathcal{MS}$, we have the classical local systems of conformal blocks $\langle \otimes_{A^0} \ \{L_a\} \rangle_{\mathcal{MS}}$ on $\mathcal{M}_{A;\lambda}$.

16.4. **Corollary.** *The local systems $\langle \otimes_{A^0} \ \{L_a\} \rangle_{\mathcal{MS}}$ are semisimple. They carry a canonical non-degenerate Hermitian form.*

In fact, the local system $f_A^* \langle \otimes_{A^0} \ \{L_a\} \rangle_{\mathcal{MS}}$ is semisimple by the previous theorem and by Beilinson-Bernstein-Deligne-Gabber, [BBD] 6.2.5. The map of fundamental groupoids $f_{A*} : \widetilde{\mathcal{T}eich}_{A'} \longrightarrow \widetilde{\mathcal{T}eich}_A$ is surjective; therefore the initial local system is semisimple.

The Hermitian form is defined in the same manner as in genus zero, cf. 11.7.

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